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INTRODUCTION TO THE

CALCULUS

THE MACMILLAN COMPANY
NEW YORK • CHICAGO
DALLAS • ATLANTA • SAN FRANCISCO
LONDON • MANILA

THE MACMILLAN COMPANY
OF CANADA, LIMITED
TORONTO

INTRODUCTION TO THE CALCULUS

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New York

THE MACMILLAN COMPANY

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Set up and electrotyped. Published January, 1921.
Published under new title, with seven new chapters, November, 1922.

Fifteenth Printing 1954

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• PRINTED IN THE UNITED STATES OF AMERICA •

PREFACE

THE present book is a revision of the author's *A First Course in the Differential and Integral Calculus.* The plan of treatment is essentially the same, but the presentation is fuller, and the lists of exercises have been enlarged by problems of value to the student of good average ability.

The object of the book is two-fold; namely, to set forth the application of the calculus to problems of geometry and physics of the first order of importance, and to make clear the thought which underlies the calculus.

To attain the first end, the physical picture must be shown to the student who has no technical knowledge of physics, but who can understand the simplest concepts of that science when clearly presented to him. Consequently, great care has been taken each time that a new physical notion has been introduced to say exactly what is meant, and then to show precisely how mathematics applies to the situation in hand.

On the other hand, thorough training in the formal part of the calculus is essential if the student is to develop power in the use of his tools, and exercises adequate for this purpose have been included in lists properly graded in point of difficulty.

Behind and beneath it all is the idea of the limit. Abstract discussions of this idea are not in place in an elementary treatment. The beginner comes to assimilate the method of limits by seeing it applied, with such details as have a meaning for him, in proving the few fundamental theorems on which the calculus rests, and in formulating geometrical and physical problems.

The treatment is flexible from start to finish. The teacher can go as far, or stop as early, as he pleases in presenting the material of a given chapter. Thus in the chapter on Definite Integrals any reasonable selection from the topics there treated can be made, the order changed, and whole paragraphs omitted without marring the unity of the course. The same is true of the chapters on Mechanics and Infinite Series. Many of these abridged treatments are altogether admirable; but no one of them can be expected to appear to any large body of teachers as preeminently the best. It is primarily a question of the personal equation of the teacher himself. The book also takes account of the personal equation of the student. A skilful teacher will help his best students to see as far and as deeply as their talents permit. He can do this with this text without losing by the way the less gifted students; for each time that the scene changes the new subject is presented with the utmost simplicity.

The book is intended alike for the engineer or the physicist and for the student of pure mathematics. The best methods of the present day in the calculus, when properly presented, are within the reach of the former student and afford him most valuable tools for the understanding of his own technical problems. On the other hand, the student of pure mathematics cannot do better than early to inform himself concerning those relations of the calculus to physics, to which this great branch of mathematics owes its origin.

Cambridge, Massachusetts,
September 27, 1922.

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CHAPTER I

INTRODUCTION

THE Calculus was invented in the seventeenth century by the mathematician, astronomer, and physicist, Sir Isaac Newton in England, and the philosopher Leibniz in Germany. The reaction of the invention on geometry and mathematical physics was most important. In fact, by far the greatest part of the mathematics and the physics of the present day owes its existence to this invention.

1. Functions. The word *function*, in mathematics, was first applied to an expression involving one or more letters which represent variable quantities; as, for example, the expressions

(a) $$x^3, \qquad 2x^3 - 3x + 1;$$

(b) $$\sqrt{x}, \qquad \sqrt{a^2 - x^2};$$

(c) $$\frac{x^2}{a+x}, \qquad \frac{xy}{x^2+y^2}, \qquad \frac{ax+by}{\sqrt{x^2+y^2+z^2}};$$

(d) $$\sin x, \qquad \log x, \qquad \tan^{-1} x.$$

In the second example under (b), two letters enter; but a is thought of as chosen in advance and then held fast, x alone being variable. A quantity of this kind is called a *constant*. Thus

$$ax + b$$

is a function of x which depends on two constants, a and b.

Such expressions are written in symbolic, or abbreviated, form as $f(x)$, $f(x, y)$ (read: "f of x," "f of x and y" etc.); other letters in common use being F, ϕ, Φ, etc.* Thus the equation

$$f(x) = 2x^3 - 3x + 1 \tag{1}$$

defines the function $f(x)$ in the present case to be $2x^3-3x+1$. Again,

$$\phi(x, y, z) = x^2 + y^2 + z^2 \tag{2}$$

is an equation defining the function $\phi(x, y, z)$ as $x^2 + y^2 + z^2$.

We shall be concerned for the present with functions of one single variable, as illustrated by (1) above. Here, x is called the *independent* variable, since we assign to it any value we like. The value of the function, or more briefly, the *function*, is called the *dependent variable*, and is often denoted by a single letter, as

$$y = f(x)$$

or

$$y = 2x^3 - 3x + 1.$$

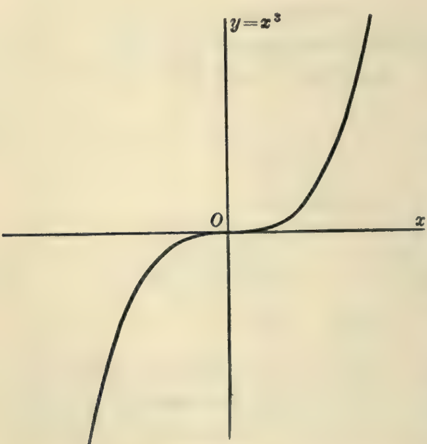


FIG. 1

Graphs. A function of a single variable,

$$y = f(x),$$

can be represented geometrically by its graph, and this representation is of great aid in studying the properties of the function. The independent variable is laid off as the x-coordinate, or abscissa, and the dependent variable, or func-

* To distinguish between $f(x)$ and $F(x)$, read the first "small f of x" and the second, "large F of x."

tion, as the y-coordinate, or ordinate. Thus the graph of the function

$$f(x) = x^3$$

is the curve

$$y = x^3.$$

Illustrations from Geometry and Physics. The familiar formulas of geometry and physics afford simple examples of functions. Thus the area, A, of a circle is given by the formula

$$A = \pi r^2,$$

where r denotes the radius, π being the fixed number 3.1416. Here, r is thought of as the independent variable, — it may have any positive value whatever, — and A is the function, or dependent variable.

Again, for the three round bodies, the volumes are:

(*a*) $V = \frac{4}{3}\pi r^3,$ sphere;

(*b*) $V = \pi r^2 h,$ cylinder;

(*c*) $V = \frac{\pi}{3} r^2 h,$ cone.

In (*b*) and (*c*), h denotes the altitude and r, the radius of the base; V is here a function of the *two* independent variables, r and h.

The surfaces of these bodies are given by the formulas:

(α) $S = 4\pi r^2,$ sphere;

(β) $S = 2\pi rh,$ cylinder;

(γ) $S = \pi rl,$ cone;

l, in the last formula, denoting the slant height. Thus we have three further examples of functions of one or of two variables.

The formula for a freely falling body is

$$s = \tfrac{1}{2} gt^2,$$

where s denotes the distance fallen and t the time; g is a constant, for it has just one value after the units of time and

length have been chosen. Here, t is the independent variable and s is the function. If, however, we solve this equation for t:

$$t=\sqrt{\frac{2s}{g}},$$

then s becomes the independent variable and t, the function.

Sometimes two variables are connected by an equation, as

$$pv=c,$$

where p denotes the pressure of a gas and v its volume, the temperature remaining constant. Here, either variable can be chosen as the independent variable, and when the equation is solved for the other variable, the latter becomes the dependent variable, or function. Thus, if we write

$$v=\frac{c}{p},$$

p is the independent variable, and v is expressed as a function of p. But if we write

$$p=\frac{c}{v},$$

the rôles are reversed.

The Independent Variable Restricted. Often the independent variable is restricted to a certain interval, as in the case of the function

$$y=\sqrt{a^2-x^2}.$$

Here, x must lie between $-a$ and a:

$$-a \leqq x \leqq a,$$

since other values of x make a^2-x^2 negative, and the above expression has no meaning.

This was also the case with the geometric examples above cited. There, r, h, l were necessarily positive, since there is no such thing, for example, as a sphere of zero or negative radius.

The independent variable may also be restricted to being a positive whole number, as in the case of the sum of the first n

terms of a geometric progression:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Here,

$$s_n = \frac{a - ar^n}{1 - r}.$$

Suppose $a = 1$, $r = \frac{1}{2}$, the progression thus becoming

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}.$$

Then

$$s_n = \frac{1 - \dfrac{1}{2^n}}{1 - \dfrac{1}{2}} = 2 - \frac{1}{2^{n-1}},$$

and we have an example of a function with the independent variable a natural number, *i.e.* a positive integer.

In the case of the functions treated in the calculus, the domain of the independent variable is a *continuum, i.e.*, for functions of a single variable, an interval, as

$$a \leqq x \leqq b, \qquad \text{or} \qquad 0 < x.$$

Ordinarily, the later letters of the alphabet, particularly x, y, z, are used to represent variables, the early letters denoting constants. Thus it will be understood, when such an expression as

$$ax^2 + bx + c$$

is written down, that a, b, c are constants and x is the variable.

Multiple-Valued Functions; Principal Value. The expressions above cited are all examples of *single-valued* functions; *i.e.* to each value of the independent variable x corresponds but one value of the function. A function may, however, be *multiple-valued;* as in the case of the function y defined by the equation

$$x^2 + y^2 = a^2.$$

Here

$$y = \pm\sqrt{a^2 - x^2},$$

and so is a double-valued function. This function is, however, completely represented by means of the two single-valued functions,

$$y = \sqrt{a^2 - x^2} \quad \text{and} \quad y = -\sqrt{a^2 - x^2}.$$

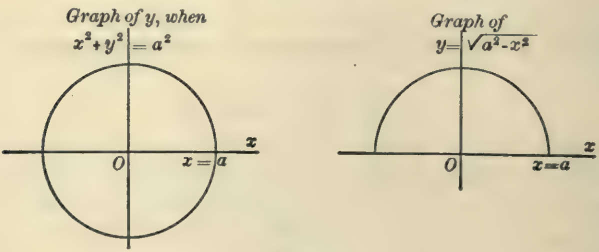


FIG. 2

They form the *branches* of this multiple-valued function.

The student should notice that the radical sign $\sqrt{}$ is defined as meaning the *positive* square root, NOT *either* the positive or the negative square root at pleasure. If it is desired to express the negative square root, the minus sign must be written in front of the radical sign, $-\sqrt{}$. Thus $\sqrt{4} = 2$, and not -2. This does not mean that 4 has only one square root. It means that the notation $\sqrt{4}$ calls for the positive, and not for the negative, of these two roots.

Again,

$$\sqrt{(-2)^2} = 2,$$

and not -2. For $(-2)^2 = 4$, and $\sqrt{}$ means the positive root. And, generally,

$$(1) \qquad \begin{cases} \sqrt{x^2} = x, & \text{if } x \text{ is positive;} \\ \sqrt{x^2} = -x, & \text{if } x \text{ is negative.} \end{cases}$$

A similar remark applies to the symbol $\sqrt[2n]{}$, which is likewise used to mean the positive $2n$th root. Moreover,

$$a^{\frac{1}{2}} = \sqrt{a}, \qquad a^{\frac{1}{2n}} = \sqrt[2n]{a}.$$

The function

$$y = \sqrt{x}$$

is often called the *principal value* of the double-valued function defined by the equation

$$y^2 = x.$$

Since multiple-valued functions are studied by means of single-valued functions, it will be understood henceforth, unless the contrary is explicitly stated, that the word *function* means *single-valued* function.

Absolute Value. It is frequently desirable to use merely the *numerical*, or *absolute value* of a quantity, and to have a notation for the same. The notation is: $|x|$, read "absolute value of x." Thus

$$|-3| = 3 \qquad \text{and} \qquad |3| = 3.$$

We can now write in a single formula what was formerly stated by the two equations (1), namely the definition of the radical sign, $\sqrt{\ }$:

$$\sqrt{a^2} = |a|. \tag{2}$$

Again, by the *difference* of two numbers we often mean the value of the larger less the smaller. Thus the difference of 4 and 10 is 6; and the difference of 10 and 4 is also 6. The difference of a and b, in this sense, can be expressed as either

$$|b - a| \qquad \text{or} \qquad |a - b|.$$

Continuous Functions. A function, $f(x)$, is said to be *continuous* if a slight change in x produces but a slight change in the value of the function. Thus the polynomials are readily shown to be continuous; cf. Chap. II, § 5, and all the functions with which we shall have to deal are continuous, save at exceptional points.

As an example of a function which is discontinuous at a certain point may be cited the function (see Fig. 3)

$$f(x) = \frac{1}{x}.$$

When x approaches the value 0, the function increases numerically without limit. The graph of the function has the axis of y as an asymptote.

The fractional rational functions are continuous except at the points at which the denominator vanishes.

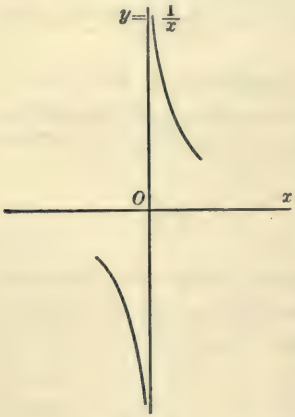


FIG. 3

Thus the function

$$f(x)=\frac{x^2+1}{x^2-1}$$

is continuous except at the points $x=1$ and $x=-1$. Here, the function becomes infinite. Its graph is the curve

$$y=\frac{x^2+1}{(x-1)(x+1)},$$

which evidently has the lines $x=1$ and $x=-1$ as asymptotes.

The function

$$f(x)=\tan x$$

is continuous except when x is an odd multiple of $\pi/2$,

$$x=\frac{2n+1}{2}\pi.$$

EXERCISES

1. If $$f(x)=x^2-4x+3,$$
show that $f(1)=0,\qquad f(2)=-1,\qquad f(3)=0.$
Compute $f(0), f(4)$. Plot the graph of the function.

2. If $$\phi(x)=4x^3$$
compute $\phi(2)$ and $\phi(\sqrt{3})$.

3. If $$F(x)=\frac{2x-3}{x+7},$$
compute $F(\sqrt{2})$ correct to three significant figures.

Ans. $-.0204$

4. If $$\Phi(x) = (x^3 - x)\sin x,$$
find all the values of x for which
$$\Phi(x) = 0.$$

5. If $$\psi(x) = x^{\frac{2}{3}} - x^{-\frac{2}{3}},$$
find $\psi(8)$.

6. Solve the equation
$$x^3 - xy + 3 = 5y$$
for y, thus expressing y as a function of x.

7. If $$f(x) = a^x,$$
show that $$f(x)f(y) = f(x+y).$$

8. If $$y = \frac{x+1}{2x-3},$$
express x as a function of y.

9. Draw the graph of the function
$$f(x) = x^2 + 4x + 3,$$
taking 1 cm. as the unit.

Suggestion: Write the function in the form, $(x+1)(x+3)$.

10. Draw the graph of the function
$$f(x) = x^3 - 4x.$$

11. Draw the graph of the function
$$f(x) = \frac{x}{x^2 - 4},$$
and hence illustrate the two discontinuities which this function has.

12. Draw the graph of the function
$$f(x) = \frac{1}{x^2} - \frac{1}{(x-1)^3}.$$

13. For what values of x are the following functions discontinuous?

$$(a)\ f(x) = \cot x; \qquad (c)\ f(x) = \csc x;$$

$$(b)\ f(x) = \sec x; \qquad (d)\ f(x) = \tan \frac{x}{2}.$$

14. Express the double-valued function defined by the equation

$$x^2 - y^2 = -1$$

in terms of two single-valued functions.

15. Express the quadruple-valued function defined by the equation

$$y^4 - 2y^2 + x^2 = 0$$

in terms of four single-valued functions.

16. Express the sum s_n of the first n terms of the arithmetic progression

$$a + (a + b) + (a + 2b) + \cdots + (a + \overline{n-1}b)$$

as a function of n.

Thus obtain the sum of the first n positive integers as a function of n.

17. If P dollars are put at simple interest for one year at r per cent, (a) express the amount A (principal and interest) as a function of P and r. (b) Express the amount A at the end of n years, the interest being compounded annually, as a function of P, r, and n. (c) Express the amount A at the end of one year, if the interest is compounded m times in the year at equal intervals, as a function of P, r, m.

2. Continuation. General Definition of a Function. The conception of the function is broader than that of the mathematical formulas mentioned in the last paragraph. Let us state the definition in its most general form.

DEFINITION OF A FUNCTION. *The variable y is said to be a function of the variable x if there exists a law whereby, when x is given, y is determined.*

Consider, for example, a quantity of gas confined in a chamber, — for instance, the charge of the mixture of gasolene and air as it is being compressed in the cylinder of an automobile. The charge exerts at each instant a definite pressure, p, of so many pounds per square inch on the walls of the chamber, and this pressure varies with the volume, v, occupied by the charge. In the small fraction of a second under consideration, presumably but little heat is gained or lost through the walls of the chamber, and thus p is a function of v,

$$p = f(v).$$

In this case, the function is given approximately by the mathematical formula

$$p = \frac{C}{v^{1.4}},$$

where C denotes a certain constant. But that which is of first importance for our conception is not the formula, but the fact that *to each value of v there corresponds a definite value of p.* In other words, there is a definite graph of the relation between v and p. The representation of the relation by a mathematical formula is, indeed, important; but what we must first see clearly is the fact that there is a definite relation to express.

As another illustration take the curve traced out by the pen of a self-registering thermometer of the kind used at a meteorological station. The instrument consists of a cylindrical drum turned slowly by clock-work at uniform speed about a vertical axis, a sheet of paper being wound

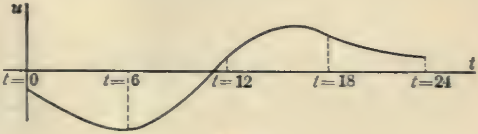


Fig. 4

firmly round the drum. A pen is held against the paper, and the height of the pen above a certain level is proportional to the height of the temperature above the temperature corre-

sponding to that level. The apparatus is set in operation, and when the drum has been turning for a day, the paper is taken off and spread out flat. Thus we have before us the graph of the temperature for the day in question, the independent variable being the time (measured in hours from midnight) and the dependent variable being the temperature, represented by the other coordinate of a point on the curve.

One more illustration, — that of the resistance of the atmosphere to a rifle bullet. This resistance, measured in pounds, depends on the velocity of the bullet, and it is a matter of physical experiment to determine the law. But that which is of first importance for our conceptions is the fact *that there is a law*, whereby, when the velocity, v, is given an arbitrary value within the limits of the velocities considered, there corresponds to this v a definite value, R, of the resistance. We say, then, that R is a *function* of v and write

$$R = f(v).$$

In this connection, cf. the chapter on Mechanics, § 7, Graph of the Resistance, in the author's *Differential and Integral Calculus.*

CHAPTER II

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS
GENERAL THEOREMS

1. Definition of the Derivative. The Calculus deals with varying quantity. If y is a function of x, then x is thought of, not as having one or another special value, but as flowing or growing, just as we think of time or of the expanding circular ripples made by a stone dropped into a placid pond. And y varies with x, sometimes increasing, sometimes decreasing. Now if we consider the change in x for a short interval, say from $x = x_0$ to $x = x'$, the corresponding change in y, as y goes from y_0 to y', will be in general almost proportional to the change in x. For the *ratio* of these changes is

$$\frac{y' - y_0}{x' - x_0},$$

and this quantity changes only slightly when x' is nearly equal to x_0. Let us study this last statement minutely.

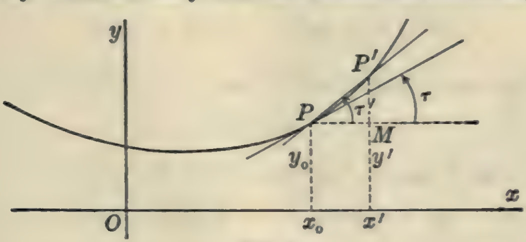


FIG. 5

The above ratio has a simple geometric meaning, if we draw the graph of the function; for

$$PM = x' - x_0; \qquad MP' = y' - y_0,$$

and
$$\frac{y' - y_0}{x' - x_0} = \tan \tau',$$

where τ' denotes the angle which the secant PP' makes with the axis of x. Now let x' approach x_0 as its limit. Then τ' approaches as its limit the angle τ which the tangent line of the graph at P makes with the axis of x, and hence

$$\lim_{x' \doteq x_0} \frac{y' - y_0}{x' - x_0} = \tan \tau$$

$\left(\text{read: "limit, as } x' \text{ approaches } x_0, \text{ of } \frac{y' - y_0}{x' - x_0}\text{"}\right)$.

The determination of this limit and the discussion of its meaning is the fundamental problem of the Differential Calculus.

Such are the concepts which underlie the idea of the derivative of a function. We turn now to a precise formulation of the definition. Let

(1) $$y = f(x)$$

be a given function of x. Let x_0 be an arbitrary value of x, and let y_0 be the corresponding value of the function:

(2) $$y_0 = f(x_0).$$

Give to x an increment,* Δx; *i.e.* let x have a new value, x', and denote the change in x, namely, $x' - x_0$, by Δx:

$$x' - x_0 = \Delta x, \qquad x' = x_0 + \Delta x.$$

The function, y, will thereby have changed to the value

(3) $$y' = f(x')$$

and hence have received an increment, Δy, where

$$y' - y_0 = \Delta y, \qquad y' = y_0 + \Delta y.$$

* The student must not think of this symbol as meaning Δ *times* x. We might have used a single letter, as h, to represent the difference in question: $x' = x_0 + h$; but h would not have reminded us that it is the increment of x, and not of y, with which we are concerned. The notation is read "*delta x*."

Equation (3) is equivalent to the following:

$$(4) \qquad y_0 + \Delta y = f(x_0 + \Delta x).$$

From equations (2) and (4) we obtain by subtraction the equation

$$\Delta y = f(x_0 + \Delta x) - f(x_0),$$

and hence

$$(5) \qquad \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Definition of a Derivative. *The limit which the ratio* (5), *namely* $\frac{\Delta y}{\Delta x}$, *approaches when* Δx *approaches zero*:

$$(6) \qquad \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} \qquad \text{or} \qquad \lim_{\Delta x \doteq 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},$$

is called the derivative of y *with respect to* x *and is denoted by* $D_x y$ *or* $D_x f(x)$ (read: "D x of y"):

$$(7) \qquad \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x y.$$

In this definition Δx may be negative as well as positive, and the limit (6) must be the same when Δx approaches 0 from the negative side as when it approaches 0 from the positive side.

To *differentiate* a function is to find its derivative.

The geometrical interpretation of the analytical process of differentiation is to find the slope of the graph of the function. For,

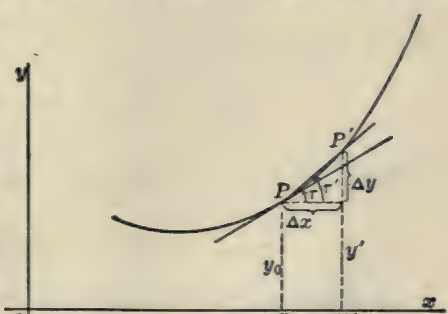


Fig. 6

$$\tan \tau' = \frac{\Delta y}{\Delta x}$$

and

$$\tan \tau = \lim_{P' \doteq P} \tan \tau' = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x y.$$

2. Differentiation of x^n. Suppose n has the value 3, so that it is required to differentiate the function

$$(1) \qquad y = x^3.$$

We must follow the definition of § 1 step by step. Begin, then, by assigning to x a particular value, x_0, which is to be held fast during the rest of the process, and compute from equation (1) the corresponding value y_0 of y:

$$(2) \qquad y_0 = x_0^3.$$

Next, give to x an arbitrary increment, Δx, denote the corresponding increment in y by Δy, and compute it. To this end we first write down the equation

$$(3) \qquad y_0 + \Delta y = (x_0 + \Delta x)^3.$$

The right-hand side of this equation can be expanded by the binomial theorem, and hence (3) can be written in a new form:*

$$(4) \qquad y_0 + \Delta y = x_0^3 + 3\,x_0^2\Delta x + 3\,x_0\Delta x^2 + \Delta x^3.$$

Subtract equation (2) from equation (4):

$$\Delta y = 3\,x_0^2\Delta x + 3\,x_0\Delta x^2 + \Delta x^3.$$

Next, divide through by Δx:

$$\frac{\Delta y}{\Delta x} = 3\,x_0^2 + 3\,x_0\Delta x + \Delta x^2.$$

We are now ready to let Δx approach 0 as its limit:

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} (3\,x_0^2 + 3\,x_0\Delta x + \Delta x^2).$$

* It is at this point that the specific properties of the function x^3 come into play. Here, it is the binomial theorem that enables us ultimately to compute the limit. In the differentiations of later paragraphs and chapters it will always be some characteristic property of the function in hand which will make possible a transformation at this point.

The limit of the left-hand side is, by definition, D_xy. On the right-hand side, each of the last two terms in the parenthesis approaches the limit 0, and so their sum approaches 0, also.

The first term does not change with Δx. Hence, the whole parenthesis approaches the limit $3x_0^2$. We have, then, as the final result:

$$D_xy = 3x_0^2.$$

The subscript has now served its purpose, which was, to remind us that x_0 is not to vary with Δx, and it may be dropped. Thus

$$D_x x^3 = 3x^2.$$

The differentiation of the function x^n in the general case that n is any positive integer can be carried through in precisely the same manner. As the result of the first step we have

$$y_0 = x_0^n. \tag{5}$$

Next comes:

$$y_0 + \Delta y = (x_0 + \Delta x)^n, \tag{6}$$

and we now apply the binomial theorem to the expression on the right-hand side. Thus

$$y_0 + \Delta y = x_0^n + nx_0^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x_0^{n-2}\Delta x^2 + \cdots + \Delta x^n. \tag{7}$$

On subtracting (5) from (7) we have:

$$\Delta y = nx_0^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x_0^{n-2}\Delta x^2 + \cdots + \Delta x^n.$$

Now divide through by Δx:

$$\frac{\Delta y}{\Delta x} = nx_0^{n-1} + \frac{n(n-1)}{1 \cdot 2}x_0^{n-2}\Delta x + \cdots + \Delta x^{n-1},$$

and let Δx approach the limit zero:

$$\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0}\left(nx_0^{n-1} + \frac{n(n-1)}{1 \cdot 2}x_0^{n-2}\Delta x + \cdots + \Delta x^{n-1}\right).$$

Each term of the parenthesis after the first is the product of a constant factor and a positive power of Δx. This second

factor approaches zero when Δx approaches zero; consequently the whole term approaches zero. There is only a fixed number of these terms, and so the whole parenthesis approaches the limit nx_0^{n-1}. Hence

$$D_x y = nx_0^{n-1}.$$

On dropping the subscript we obtain the final result:

$$D_x x^n = nx^{n-1}. \tag{8}$$

In particular, if $n = 1$, we have

$$D_x x = 1. \tag{9}$$

EXERCISES

Differentiate the following seven functions, applying the process of § 1 step by step.

1. $y = 4x^3$. *Ans* $D_x y = 12x^2$.
2. $y = x^4$.
3. $y = 2x^2 - 3x + 1$. *Ans.* $D_x y = 4x - 3$.
4. $y = x^7 - x^5$. *Ans.* $D_x y = 7x^6 - 5x^4$.
5.* $f(x) = 1 - 2x^4$. *Ans.* $D_x f(x) = -8x^3$.
6. $\phi(x) = x^2 - 2x + 1$.
7. $F(x) = (1 - x)^2$.
8. Let $y = 5x - x^2$,

and take $x_0 = 1$; then $y_0 = 4$. If $\Delta x = .2$, then $\Delta y = .56$ and $\frac{\Delta y}{\Delta x} = 2.8$. Show further that,

for $\Delta x = .1$, $\Delta y = .29$, $\frac{\Delta y}{\Delta x} = 2.9$;

and

for $\Delta x = .01$, $\Delta y = .0299$, $\frac{\Delta y}{\Delta x} = 2.99$.

* It is immaterial whether we write

$$y = 1 - 2x^4 \qquad \text{or} \qquad f(x) = 1 - 2x^4.$$

Plot the curve accurately for values of x from $x=0$ to $x=5$, taking 1 cm. as the unit, and draw the secants * in each of the three foregoing cases.

What appears to be the slope of the curve at the point $(x_0, y_0)=(1, 4)$? Prove your guess to be correct.

9. In Ex. 7, let $x_0=-1$. If Δx is given successively the values .01 and $-.01$, compute Δy and $\frac{\Delta y}{\Delta x}$.

10. Complete the following table:

Δx	Δy	$\tan\tau' = \frac{\Delta y}{\Delta x}$
.1		
.01		
.001		

for each of the functions:

(a) $y=x^2-2x+1, \qquad x_0=2;$

(b) $y=x-x^3, \qquad x_0=-1;$

(c) $y=3x^2-x, \qquad x_0=0.$

11. By means of the general theorem (8) write down the derivatives of the following functions:

$$x^4; \qquad x^5; \qquad x^{10}; \qquad x; \qquad x^{99}.$$

By means of the definition of § 1 differentiate each of the following functions:

12. $y=\frac{1}{x}.$ *Ans.* $D_x y=-\frac{1}{x^2}.$

13. $y=\frac{1}{x^2}.$ *Ans.* $D_x y=-\frac{2}{x^3}.$

14. $y=\frac{1}{x^3}.$ *Ans.* $D_x y=-\frac{3}{x^4}.$

* The student should recall from his earlier work how to draw a straight line on squared paper when a point and the slope of the line are given.

3. Derivative of a Constant. The function

$$f(x) = c,$$

where c denotes a constant, has for its graph a right line parallel to the axis of x. Since the derivative of a function is represented geometrically by the slope of its graph, it is clear that the derivative of this function is zero:

$$D_x c = 0.$$

It is instructive, however, to obtain this result analytically by the process of § 1. We have here:

$$y_0 = f(x_0) = c,$$

$$y_0 + \Delta y = f(x_0 + \Delta x) = c;$$

hence $$\Delta y = 0 \qquad \text{and} \qquad \frac{\Delta y}{\Delta x} = 0.$$

Now allow Δx to approach 0. The value of $\Delta y / \Delta x$ is always 0, and hence its limit* is 0:

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 0, \qquad \text{or} \qquad D_x c = 0.$$

* We note here an error frequently made in presenting the subject of limits in school mathematics. It is there often stated that "a variable X approaches a limit A if X comes indefinitely near to A, but *never reaches* A." This last requirement is not a part of the conception of a variable's approaching a limit. It is true that it is often inexpedient to allow the *independent variable* to reach its limit. Thus, in differentiating a function, the ratio $\Delta y / \Delta x$ ceases to have a meaning when $\Delta x = 0$, since division by 0 is impossible. The problem of differentiation is not to find the value of $\Delta y / \Delta x$ when $\Delta x = 0$; such a question would be absurd. What we do is to allow Δx to approach zero as its limit without ever reaching that limit. We can do this for the reason that Δx is the *independent variable.*

When, however, it is Δy or $\Delta y / \Delta x$ that is under consideration, we have to do with *dependent variables*, and we have no control over them, as to whether they reach their limit or not. Thus in the case of the text both Δy and $\Delta y / \Delta x$ are constants ($= 0$). When Δx approaches 0, they always have one and the same value, and so, under the correct conception of approach to a limit each approaches a limit, namely 0.

We can state the result by saying: *The derivative of a constant is* 0.

4. Differentiation of $\sqrt{x}$. Let us differentiate

$$y=\sqrt{x}.$$

Here, $\quad y_0=\sqrt{x_0}, \qquad y_0+\Delta y=\sqrt{x_0+\Delta x},$

$$\frac{\Delta y}{\Delta x}=\frac{\sqrt{x_0+\Delta x}-\sqrt{x_0}}{\Delta x}.$$

We cannot as yet see what limit the right-hand side approaches when Δx approaches 0, for both numerator and denominator approach 0, and $\frac{0}{0}$ has no meaning. We can, however, transform the fraction by multiplying numerator and denominator by the *sum* of the radicals and recalling the formula of Elementary Algebra:

$$a^2-b^2=(a-b)(a+b).$$

Thus $$\frac{\Delta y}{\Delta x}=\frac{\sqrt{x_0+\Delta x}-\sqrt{x_0}}{\Delta x}\cdot\frac{\sqrt{x_0+\Delta x}+\sqrt{x_0}}{\sqrt{x_0+\Delta x}+\sqrt{x_0}}$$

$$=\frac{1}{\Delta x}\cdot\frac{(x_0+\Delta x)-x_0}{\sqrt{x_0+\Delta x}+\sqrt{x_0}}=\frac{1}{\sqrt{x_0+\Delta x}+\sqrt{x_0}},$$

and hence $$\lim_{\Delta x\doteq 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\doteq 0}\frac{1}{\sqrt{x_0+\Delta x}+\sqrt{x_0}}=\frac{1}{2\sqrt{x_0}}.$$

Dropping the subscript, we have:

$$D_x\sqrt{x}=\frac{1}{2\sqrt{x}}.$$

EXERCISES

1. Differentiate the function $y=\frac{1}{\sqrt{x}}$. *Ans.* $D_xy=-\frac{1}{2\sqrt{x^3}}$.

2. If $$y=\sqrt{2-3x},$$

show that $$D_xy=\frac{-3}{2\sqrt{2-3x}}.$$

3. Prove: $$D_x\sqrt{1-x} = -\frac{1}{2\sqrt{1-x}}.$$

4. Prove: $$D_x\sqrt{a+bx} = \frac{b}{2\sqrt{a+bx}}.$$

5. Three Theorems about Limits. Infinity.* In the further treatment of differentiation the following theorems are needed.

THEOREM I. *The limit of the sum of two variables is equal to the sum of their limits:*

$$\lim(X+Y) = \lim X + \lim Y.$$

In this theorem we think of X and Y as two *dependent variables*, each of which approaches a limit:

$$\lim X = A, \qquad \lim Y = B.$$

We do not care what the independent variable may be. In the applications of the theorem to computing derivatives, the independent variable will always be Δx, and it will be allowed to approach 0, without ever reaching its limit.

Since X approaches A, it comes nearer and nearer to this value. Let the difference between the variable and its limit be denoted by ϵ; then the limit of ϵ is 0:

(1) $$X - A = \epsilon, \qquad X = A + \epsilon;$$

$$\lim \epsilon = 0.$$

Similarly, let

(2) $$Y - B = \eta, \qquad Y = B + \eta;$$

then $$\lim \eta = 0.$$

* This paragraph should be read carefully and its content grasped, but the student should not be required to reproduce it at this stage of his work. He will meet frequent applications of its principles, and he should turn back each time to these pages and read anew the theorem involved, with its proof. When he has thus come to see the full meaning and importance of these theorems, he should demand of himself that he be able readily to reproduce the proofs.

It will be convenient to think of these numbers as represented geometrically by points on the scale of numbers, thus:

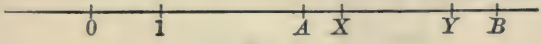


Fig. 7

Of course, A and B may be negative or 0. ϵ and η may be negative as well as positive, or even 0.

Consider the variable $X+Y$. Its value from (1) and (2) is:

$$X+Y=A+B+\epsilon+\eta.$$

Hence $$\lim(X+Y)=\lim(A+B+\epsilon+\eta).$$

But since $\lim \epsilon = 0$ and $\lim \eta = 0$, the limit of the right-hand side of this equation is $A+B$, or

$$\lim(X+Y)=A+B.$$

Consequently, $$\lim(X+Y)=\lim X+\lim Y, \qquad \text{q. e. d.}$$

Corollary. *The limit of the sum of any fixed number of variables is equal to the sum of the limits of these variables:*

$$\lim(X_1+X_2+\cdots+X_n)=\lim X_1+\lim X_2+\cdots+\lim X_n.$$

Suppose $n=3$. Then

$$X_1+X_2+X_3=(X_1+X_2)+X_3.$$

From Theorem I it follows that

$$\lim(X_1+X_2+X_3)=\lim(X_1+X_2)+\lim X_3.$$

Applying the Theorem again, we have

$$\lim(X_1+X_2)=\lim X_1+\lim X_2.$$

Hence the corollary is true for $n=3$. It can now be established for $n=4$; and so on. By the method of Mathematical Induction it can be proven generally. Or, the proof of the main theorem may be extended directly to the present theorem.

THEOREM II. *The limit of the product of two variables is equal to the product of their limits:*

$$\lim (XY) = (\lim X)(\lim Y).$$

From equations (1) and (2) it follows that

$$XY = (A + \epsilon)(B + \eta),$$

or
$$XY = AB + B\epsilon + A\eta + \epsilon\eta.$$

Hence
$$\lim XY = \lim (AB + B\epsilon + A\eta + \epsilon\eta).$$

Since A and B are constants, each of the last three terms in the parenthesis approaches the limit 0, and so the limit of the parenthesis is AB. Hence

$$\lim (XY) = AB,$$

or
$$\lim (XY) = (\lim X)(\lim Y), \qquad \text{q.e.d.}$$

COROLLARY. *The limit of the product of n variables is equal to the product of the limits of these variables:*

$$\lim (X_1 X_2 \cdots X_n) = (\lim X_1)(\lim X_2) \cdots (\lim X_n).$$

The proof is similar to that of the corollary under Theorem I.

Remark. As a particular case under Theorem II we have:

$$\lim (CX) = C(\lim X),$$

where C is a constant.

THEOREM III. *The limit of the quotient of two variables is equal to the quotient of their limits, provided that the limit of the divisor is not 0:*

$$\lim \frac{X}{Y} = \frac{\lim X}{\lim Y}, \qquad \text{if } \lim Y \neq 0.$$

From equations (1) and (2) above we have:

$$\frac{X}{Y} = \frac{A + \epsilon}{B + \eta}.$$

Subtract A/B from each side of this equation and reduce:

$$\frac{X}{Y}-\frac{A}{B}=\frac{A+\epsilon}{B+\eta}-\frac{A}{B}=\frac{B\epsilon-A\eta}{B^2+B\eta}.$$

Hence $$\frac{X}{Y}=\frac{A}{B}+\frac{B\epsilon-A\eta}{B^2+B\eta},$$

and $$\lim\frac{X}{Y}=\lim\left(\frac{A}{B}+\frac{B\epsilon-A\eta}{B^2+B\eta}\right).$$

We wish to show that

$$\lim\frac{B\epsilon-A\eta}{B^2+B\eta}=0.$$

The numerator is seen at once to approach zero. The limit of the denominator is B^2. Let H be a positive number less than

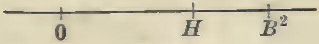


FIG. 8

B^2 Then the denominator will finally become and remain greater than H, and hence the numerical value of the quotient in question will not exceed the numerical value of

$$\frac{B\epsilon-A\eta}{H}.$$

But the limit of this expression is zero, and hence

$$\lim\frac{X}{Y}=\frac{A}{B},$$

or $$\lim\frac{X}{Y}=\frac{\lim X}{\lim Y}, \qquad \text{q. e. d.}$$

In particular, we see that, *if a variable approaches unity as its limit, its reciprocal also approaches unity:*

If $\lim X=1$, *then* $\lim\frac{1}{X}=1.$

Also,
$$\lim \frac{C}{X} = \frac{C}{\lim X},$$
where C is a constant and $\lim X \neq 0$.

Remark. If the denominator Y approaches 0 as its limit, no general inference about the limit of the fraction can be drawn, as the following examples show. Let Y have the values:
$$Y = \frac{1}{10}, \ \frac{1}{100}, \ \frac{1}{1000}, \cdots, \ \frac{1}{10^n}, \cdots.$$

(1) If the corresponding values of X are:
$$X = \frac{1}{10^2}, \ \frac{1}{100^2}, \ \frac{1}{1000^2}, \cdots, \ \frac{1}{10^{2n}}, \cdots,$$
then
$$\lim \frac{X}{Y} = \lim \frac{1}{10^n} = 0.$$

(2) If
$$X = \frac{1}{\sqrt{10}}, \ \frac{1}{\sqrt{100}}, \ \frac{1}{\sqrt{1000}}, \cdots, \ \frac{1}{10^{\frac{n}{2}}}, \cdots,$$
then $X/Y = 10^{n/2}$ approaches no limit, but increases beyond all limit.

(3) If
$$X = \frac{c}{10}, \ \frac{c}{100}, \ \frac{c}{1000}, \cdots, \ \frac{c}{10^n}, \cdots,$$
where c is any arbitrarily chosen fixed number, then
$$\lim \frac{X}{Y} = c.$$

(4) If
$$X = \frac{1}{10}, \ -\frac{1}{100}, \ \frac{1}{1000}, \ -\frac{1}{10{,}000}, \cdots,$$
then X/Y assumes alternately the values $+1$ and -1, and hence, although remaining finite, approaches no limit.

To sum up, then, we see that when X and Y both approach 0 as their limit, their ratio may approach any limit whatever, or it may increase beyond all limit, or finally, although *remain-*

ing finite, i.e. always lying between two *fixed* numbers, no matter how widely the latter may differ from each other in value, — it may jump about and so fail to approach a limit.

Infinity. If $\lim X = A \neq 0$ and $\lim Y = 0$, then X/Y increases beyond all limit, or *becomes infinite.* A variable Z is said to become infinite when it ultimately becomes and remains greater numerically than any preassigned quantity, however large.* If it takes on only positive values, it *becomes positively infinite;* if only negative values, it *becomes negatively infinite.* We express its behavior by the notation:

$$\lim Z = \infty \quad \text{or} \quad \lim Z = +\infty \quad \text{or} \quad \lim Z = -\infty.$$

But this notation does not imply that infinity is a limit; the variable in this case approaches no limit. And so the notation should not be read "Z approaches infinity" or "Z equals infinity; but "*Z becomes infinite.*"

Thus if the graph of a function has its tangent at a certain point parallel to the axis of ordinates, we shall have for that point:

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \infty;$$

read: "$\Delta y / \Delta x$ becomes infinite when Δx approaches 0."

Some writers find it convenient to use the expression "a variable approaches a limit" to include the case that the variable becomes infinite. We shall not adopt this mode of expression, but shall understand the words "approaches a limit" in their strict sense.

If a function $f(x)$ becomes infinite when x approaches a certain value a, as for example

$$f(x) = \frac{1}{x} \quad \text{for} \quad a = 0,$$

* Note that the statement sometimes made that "Z becomes greater than any assignable quantity" is absurd. There is no quantity that is greater than any assignable quantity.

we denote this by writing

$$f(a)=\infty$$

(or $f(a)=+\infty$ or $=-\infty$, if this happens to be the case and we wish to call attention to the fact).

It is in this sense that the equation

$$\tan 90^\circ=\infty$$

is to be understood in Trigonometry. The equation does *not* mean that 90° *has* a tangent and that the *value* of the latter is ∞. It means that, as x approaches 90° as its limit, $\tan x$ exceeds numerically any number one may name in advance, and stays above this number as x continues to approach 90° without ever reaching its limit, 90°.

Definition of a Continuous Function. We can now make more explicit the definition given in Chapter I by saying: $f(x)$ is continuous at the point $x=a$ if

$$\lim_{x\doteq a} f(x)=f(a).$$

From Exercises 1–3 below it follows that the polynomials are continuous for all values of x, and that the fractional rational functions are continuous except when the denominator vanishes.

EXERCISES

1. Show that, if n is any positive integer,

$$\lim (X^n)=(\lim X)^n.$$

2. If $$G(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n,$$

then $$\lim_{x\doteq a} G(x)=G(a)=c_0+c_1a+c_2a^2+\cdots+c_na^n.$$

3. If $G(x)$ and $F(x)$ are any two polynomials and if $F(a)\neq 0$

then $$\lim_{x\doteq a}\frac{G(x)}{F(x)}=\frac{G(a)}{F(a)}.$$

4. If X remains finite and Y approaches 0 as its limit, show that

$$\lim (XY) = 0.$$

5. Show that
$$\lim_{x=\infty} \frac{x^2+1}{3x^2+2x-1} = \frac{1}{3}.$$

Suggestion. Begin by dividing the numerator and the denominator by x^2.

Evaluate the following limits:

6. $\displaystyle\lim_{x=\infty} \frac{x+1}{x^3-7x+3}.$

7. $\displaystyle\lim_{x=\infty} \frac{12x^6+5}{4x^6+3x^4+7x^2-1}.$

8. $\displaystyle\lim_{x=\infty} \frac{ax+bx^{-1}}{cx+dx^{-1}}.$

9. $\displaystyle\lim_{x=0} \frac{ax+bx^{-1}}{cx+dx^{-1}}.$

10. $\displaystyle\lim_{x=-\infty} \frac{\sqrt{1+x^2}}{x}.$

11. $\displaystyle\lim_{x=\infty} \frac{x^2+x+1}{\sqrt{3+5x^2+4x^4}}.$

12. $\displaystyle\lim_{x=\infty} \frac{\sqrt{1+x^2+x^4}}{x}.$

13. $\displaystyle\lim_{x=\infty} \frac{x}{\sqrt{1+x^4}}.$

6. General Formulas of Differentiation.

THEOREM I. *The derivative of the product of a constant and a function is equal to the product of the constant into the derivative of the function:*

(I)
$$D_x(cu) = cD_xu.$$

For, let
$$y = cu.$$

Then
$$y_0 = cu_0,$$

$$y_0 + \Delta y = c(u_0 + \Delta u),$$

hence
$$\Delta y = c\Delta u,$$

$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x},$$

and
$$\lim_{\Delta x=0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x=0}\left(c\frac{\Delta u}{\Delta x}\right).$$

The limit of the left-hand side is D_xy. On the right, $\Delta u/\Delta x$ approaches D_xu as its limit. Hence by § 5, Theorem II, the limit of the right-hand side is cD_xu, and we have

$$D_x(cu) = cD_xu, \qquad \text{q. e. d.}$$

THEOREM II. *The derivative of the sum of two functions is equal to the sum of their derivatives:*

$$\text{(II)} \qquad D_x(u+v) = D_xu + D_xv.$$

For, let $$y = u + v.$$

Then $$y_0 = u_0 + v_0,$$

$$y_0 + \Delta y = u_0 + \Delta u + v_0 + \Delta v,$$

hence $$\Delta y = \Delta u + \Delta v,$$

and $$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

When Δx approaches 0, the first term on the right approaches D_xu and the second D_xv. Hence by § 5, Theorem I, the whole right-hand side approaches $D_xu + D_xv$, and we have

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0}\left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right) = \lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \doteq 0} \frac{\Delta v}{\Delta x},$$

or $$D_xy = D_xu + D_xv, \qquad \text{q. e. d.}$$

COROLLARY. *The derivative of the sum of any number of functions is equal to the sum of their derivatives.*

If we have the sum of three functions, we can write

$$u + v + w = u + (v + w).$$

Hence $$D_x(u + v + w) = D_xu + D_x(v + w)$$
$$= D_xu + D_xv + D_xw.$$

Next, we can consider the sum of four functions, and so on. Or we can extend the proof of Theorem II immediately to the sum of n functions.

Polynomials. We are now in a position to differentiate any polynomial. For example:

$$D_x(7x^4 - 5x^3 + x + 2)$$
$$= D_x(7x^4) + D_x(-5x^3) + D_x x + D_x 2$$
$$= 7D_x x^4 - 5D_x x^3 + 1 = 28x^3 - 15x^2 + 1.$$

EXERCISES

Differentiate the following functions:

1. $y = 2x^2 - 3x + 1.$ *Ans.* $D_x y = 4x - 3.$

2. $y = a + bx + cx^2.$ *Ans.* $D_x y = b + 2cx.$

3. $y = x^4 - 3x^3 + x - 1.$ *Ans.* $D_x y = 4x^3 - 9x^2 + 1.$

4. $y = a + bx + cx^2 + dx^3.$

5. $y = \dfrac{x^6 - 3x^4 - 2x + 1}{2}.$ *Ans.* $3x^5 - 6x^3 - 1.$

6. $f(x) = \dfrac{ax^2 + 2bx + c}{2h}.$ *Ans.* $\dfrac{ax + b}{h}.$

7. $\pi x^4 - 3\frac{3}{4}x^2 + \sqrt{3}.$ *Ans.* $4\pi x^3 - 7\frac{1}{2}x.$

8. Differentiate

(*a*) $v_0 t - 16t^2$ with respect to t;

(*b*) $a + bs + cs^2$ with respect to s;

(*c*) $.01ly^4 - 8.15my^2 - .9lm$ with respect to y.

9. Find the slope of the curve

$$4y = x^4 - 8x - 1$$

at the point $(1, -2)$. *Ans.* -1

10. At what angle does the curve

$$8y = 4x - x^3$$

cut the negative axis of x?

11. At what angles do the curves $y = x^2$ and $y = x^3$ intersect?
Ans. $0°$ and $8° 7'$.

12. At what angles do the curves $y = x^3 - 3x$ and $y = x$ intersect?
Ans. $26° 34'$ and $63° 26'$.

7. General Formulas of Differentiation, Continued.

THEOREM III. *The derivative of a product is given by the formula:*

$$\text{(III)} \qquad D_x(uv) = uD_xv + vD_xu.$$

Let $$y = uv.$$

Then $$y_0 = u_0v_0,$$

$$y_0 + \Delta y = (u_0 + \Delta u)(v_0 + \Delta v),$$

$$\Delta y = u_0\Delta v + v_0\Delta u + \Delta u\Delta v,$$

$$\frac{\Delta y}{\Delta x} = u_0\frac{\Delta v}{\Delta x} + v_0\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x},$$

and, by Theorem I, § 5:

$$\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0}\left(u_0\frac{\Delta v}{\Delta x}\right) + \lim_{\Delta x \doteq 0}\left(v_0\frac{\Delta u}{\Delta x}\right) + \lim_{\Delta x \doteq 0}\left(\Delta u\frac{\Delta v}{\Delta x}\right).$$

By Theorem II, § 5, the last limit has the value 0, since $\lim \Delta u = 0$ and $\lim(\Delta v/\Delta x) = D_xv$. The first two limits have the values u_0D_xv and v_0D_xu respectively.* Hence, dropping the subscripts, we have:

$$D_xy = uD_xv + vD_xu, \qquad \text{q. e. d.}$$

By a repeated application of this theorem the product of any number of functions can be differentiated. When more

* More strictly, the notation should read here, before the subscripts are dropped: $[D_xv]_{x=x_0}$, etc. Similarly in the proofs of Theorem I, II, and V.

than two factors are present, the formula is conveniently written in the form:

(1) $$\frac{D_x(uvw)}{uvw}=\frac{D_xu}{u}+\frac{D_xv}{v}+\frac{D_xw}{w}.$$

For a reason that will appear later, this is called the *logarithmic derivative* of uvw.

THEOREM IV. *The derivative of a quotient is given by the formula:**

(IV) $$D_x\left(\frac{u}{v}\right)=\frac{vD_xu-uD_xv}{v^2}.$$

Let $$y=\frac{u}{v}.$$

Then $$y_0=\frac{u_0}{v_0}, \qquad y_0+\Delta y=\frac{u_0+\Delta u}{v_0+\Delta v},$$

$$\Delta y=\frac{u_0+\Delta u}{v_0+\Delta v}-\frac{u_0}{v_0}=\frac{v_0\Delta u-u_0\Delta v}{v_0(v_0+\Delta v)},$$

$$\frac{\Delta y}{\Delta x}=\frac{v_0\dfrac{\Delta u}{\Delta x}-u_0\dfrac{\Delta v}{\Delta x}}{v_0(v_0+\Delta v)}.$$

By Theorem III of § 5 we have:

$$\lim_{\Delta x\doteq 0}\frac{\Delta y}{\Delta x}=\frac{\lim\limits_{\Delta x\doteq 0}\left(v_0\dfrac{\Delta u}{\Delta x}-u_0\dfrac{\Delta v}{\Delta x}\right)}{\lim\limits_{\Delta x\doteq 0}[v_0(v_0+\Delta v)]}.$$

Applying Theorems I and II of § 5 and dropping the subscripts we obtain:

$$D_xy=\frac{vD_xu-uD_xv}{v^2}, \qquad \text{q. e. d.}$$

* The student may find it convenient to remember this formula by putting it into words: "The denominator into the derivative of the numerator, minus the numerator into the derivative of the denominator, over the square of the denominator."

Example 1. Let
$$y=\frac{2-3x}{1-2x}.$$

Then
$$D_xy=\frac{(1-2x)D_x(2-3x)-(2-3x)D_x(1-2x)}{(1-2x)^2}$$
$$=\frac{(1-2x)(-3)-(2-3x)(-2)}{(1-2x)^2}=\frac{1}{(1-2x)^2}.$$

Example 2. To prove that the theorem
$$D_xx^n=nx^{n-1}$$
is true when n is a negative integer, $n=-m$. Here
$$x^n=\frac{1}{x^m}.$$

Hence
$$D_xx^n=\frac{x^mD_x1-1\,D_xx^m}{x^{2m}}=-\frac{mx^{m-1}}{x^{2m}}=-mx^{-m-1}.$$

On replacing m in this last expression by its value, $-n$, the proof is complete.

EXERCISES

Differentiate the following functions:

1. $$y=\frac{x}{1-x^2}.\qquad \textit{Ans.}\quad D_xy=\frac{1+x^2}{(1-x^2)^2}.$$

2. $$y=\frac{1}{1+x^2}.\qquad \textit{Ans.}\quad D_xy=\frac{-2x}{(1+x^2)^2}.$$

3. $$y=\frac{x^3}{1-x}.\qquad \textit{Ans.}\quad D_xy=\frac{3x^2-2x^3}{(1-x)^2}.$$

4. $$y=\frac{x^2}{1+x}.\qquad \textit{Ans.}\quad D_xy=\frac{2x+x^2}{(1+x)^2}.$$

5. $$s=\frac{1-t}{1+t}.\qquad \textit{Ans.}\quad D_ts=\frac{-2}{(1+t)^2}.$$

6. $$\frac{z^2+a^2}{z+a}.\qquad \textit{Ans.}\quad \frac{z^2+2az-a^2}{z^2+2az+a^2}.$$

7. $\dfrac{2ay}{a^2-y^2}$.

8. $\dfrac{ax+b}{x^2+px+q}$

9. $\dfrac{x^3+a^3}{x+a}$.

10. $\dfrac{x^2+a^2}{x^4+a^4}$.

11. $\dfrac{a+bx+cx^2}{x}$. *Ans.* $c-\dfrac{a}{x^2}$.

12. $\dfrac{3-4x+x^3}{x^2}$.

13. $\dfrac{x^6+1}{x^3}$.

8. General Formulas of Differentiation, Concluded. *Composite Functions.*

THEOREM V. *If u is expressed as a function of y and y in turn as a function of x:*

$$u=f(y), \qquad y=\phi(x),$$

then

$$D_x u = D_y u \cdot D_x y. \tag{V}$$

Here $\quad y_0=\phi(x_0), \qquad u_0=f(y_0),$

$$y_0+\Delta y=\phi(x_0+\Delta x), \qquad u_0+\Delta u=f(y_0+\Delta y),$$

$$\Delta u = f(y_0+\Delta y)-f(y_0),$$

$$\frac{\Delta u}{\Delta x}=\frac{f(y_0+\Delta y)-f(y_0)}{\Delta y}\cdot\frac{\Delta y}{\Delta x}.$$

When Δx approaches 0, Δy also approaches 0, and hence the limit of the right-hand side is

$$\left(\lim_{\Delta y \doteq 0}\frac{f(y_0+\Delta y)-f(y_0)}{\Delta y}\right)\left(\lim_{\Delta x\doteq 0}\frac{\Delta y}{\Delta x}\right)=D_y f(y) D_x y.$$

The limit of the left-hand side is $D_x u$. Consequently

$$D_x u = D_y u \cdot D_x y, \qquad \text{q. e. d.}$$

This equation can also be written in the form:

$$D_x u = D_y f(y) D_x \phi(x). \tag{V$'$}$$

The truth of the theorem does not depend on the particular letters by which the variables are denoted. We may replace, for example, x by t and y by x:

$$D_t u = D_x u \, D_t x.$$

Dividing through by the second factor on the right, we thus obtain the formula:

$$(\mathrm{V}'') \qquad D_x u = \frac{D_t u}{D_t x}.$$

Example 1. In § 4 we differentiated the function $\sqrt{x}$, and we saw that other radicals can be differentiated in a similar manner. But each new differentiation required the evaluation of $\lim \Delta y/\Delta x$ by working through the details of a limiting process. Theorem V enables us to avoid such computations, as the following example will show.

To differentiate the function

$$u = \sqrt{a^2 - x^2}.$$

Let $$y = a^2 - x^2.$$

Then $$u = \sqrt{y},$$

and the differentiation thus comes directly under Theorem V, if we set

$$f(y) = \sqrt{y}, \qquad \phi(x) = a^2 - x^2.$$

Hence we have:

$$(1) \qquad D_x u = D_y \sqrt{y} \; D_x(a^2 - x^2).$$

Now, the formula $$D_x \sqrt{x} = \frac{1}{2\sqrt{x}}$$

does not mean that the independent variable must be denoted by the letter x. If the independent variable is y, the formula reads:

$$D_y \sqrt{y} = \frac{1}{2\sqrt{y}}.$$

Consequently (1) can be written in the form:

$$(2) \qquad D_x u = \frac{1}{2\sqrt{y}}(-2x) = -\frac{x}{\sqrt{a^2-x^2}}.$$

We have, then, as the final result:

$$D_x\sqrt{a^2-x^2} = \frac{-x}{\sqrt{a^2-x^2}}.$$

Example 2. To differentiate the function

$$y = \frac{1}{(1-x)^3}.$$

Let $$z = 1 - x.$$

Then $$y = z^{-3}.$$

To apply Theorem V in the present case, the letters u and y must be replaced respectively by y and z. Thus Theorem V reads here:

$$D_x y = D_z y D_x z,$$

or $$D_x y = D_z z^{-3} D_x(1-x).$$

Since Formula (8) of § 2 has been extended to negative integral values of n by § 7, Ex. 2, we have:

$$D_z z^{-3} = -3z^{-4}.$$

Hence $$D_x y = -3z^{-4}(-1) = \frac{3}{z^4},$$

or $$D_x \frac{1}{(1-x)^3} = \frac{3}{(1-x)^4}.$$

EXERCISES

Differentiate the following functions:

1. $y = \sqrt{a^2+x^2}.$ *Ans.* $\frac{x}{\sqrt{a^2+x^2}}.$

2.* $y = \frac{1}{\sqrt{a^2-x^2}}.$ *Ans.* $\frac{x}{\sqrt{(a^2-x^2)^3}}.$

* Note that Formula (8) of § 2 has also been shown to hold for the case $n = -\frac{1}{2}$; § 4, Ex. 1.

3. $y=\sqrt{1+x+x^2}$. *Ans.* $\dfrac{1+2x}{2\sqrt{1+x+x^2}}$.

4. $y=\dfrac{1}{\sqrt{3-2x+4x^2}}$. *Ans.* $\dfrac{1-4x}{\sqrt{(3-2x+4x^2)^3}}$.

5. $u=\dfrac{x}{(1-x)^3}$. *Ans.* $\dfrac{1+2x}{(1-x)^4}$.

6. $u=\dfrac{x^2+1}{(2-3x)^2}$. *Ans.* $\dfrac{6+4x}{(2-3x)^3}$.

7. $y=\dfrac{x^2}{(1+2x)^4}$.

8. $y=\dfrac{(1-x)^3}{(2+x)^3}$.

9. $y=\left(\dfrac{x}{1-x}\right)^4$.

10. $u=\dfrac{x^2}{1-2x+x^2}$.

11.* $u=x(1-x)^4$. *Ans.* $(1-5x)(1-x)^3$.

12. $u=x(a+bx)^n$. *Ans.* $[a+(n+1)bx](a+bx)^{n-1}$.

13. $u=x^2(a+bx)^n$.

14. $u=x^3(1-x)^4$.

15. $u=x\sqrt{a-x}$. *Ans.* $\dfrac{2a-3x}{2\sqrt{a-x}}$.

16. $u=x^2\sqrt{a^2-x^2}$.

17. $u=x\sqrt{1+x+x^2}$.

18. $u=\dfrac{x}{\sqrt{a^2-x^2}}$.

19. $u=\dfrac{x}{\sqrt{1+x+x^2}}$.

20. $u=\sqrt{\dfrac{a+bx}{c+dx}}$.

21.† $u=\dfrac{1}{(a^2-2ax)^4}$.

22. $u=\dfrac{1}{(x^2-1)^2}$.

23. $u=\dfrac{1}{1+x+x^2}$.

24. $u=\dfrac{x^3-3bx^2+3b^2x-b^3}{b-x}$.

25. $u=\dfrac{a+b}{(a+bx)^2}$.

* Use Theorem III.

† Do not use Theorem IV.

9. Differentiation of Implicit Algebraic Functions. When x and y are connected by such a relation as

$$x^2 + y^2 = a^2,$$

or

$$x^3 - 2xy + y^5 = 0,$$

or

$$xy \sin y = x + y \log x,$$

i.e. if y is given as a function of x by an equation,

$$F(x, y) = 0 \qquad \text{or} \qquad \Phi(x, y) = \Psi(x, y),$$

which must first be solved for y, then y is said to be an *implicit function* of x. If we solve the equation for y, thus obtaining the equation

$$y = f(x),$$

y thereby becomes an *explicit function* of x.

By an *algebraic function* of x is meant a function y which satisfies an equation of the form

$$G(x, y) = 0,$$

where $G(x, y)$ is an *irreducible* polynomial in x and y; *i.e.* a polynomial that cannot be factored and written as the product of two polynomials.

Thus the polynomials are algebraic functions; for if

$$y = a_0 + a_1 x + \cdots + a_n x^n = P(x),$$

then y satisfies the algebraic equation

$$G(x, y) = y - P(x) = 0.$$

Similarly, the fractions in x are algebraic functions; for if

$$y = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials having no common factor, then y satisfies the algebraic equation

$$G(x, y) = Q(x)y - P(x) = 0.$$

The polynomials and the fractions are also called *rational functions.* Thus,

$$\frac{ax+by}{x^2+y^2}$$

is a rational function of the two independent variables x and y.

Again, all roots of polynomials, as

$$y=\sqrt{1+x+x^3},$$

or such functions as

$$y=\sqrt[5]{\frac{x}{1-x}}+\sqrt{4-3x^2},$$

are algebraic, as is seen on freeing the equation from radicals and transposing. The converse, however, — namely, that every algebraic function can be expressed by means of rational functions and radicals, — is not true.

In order to differentiate an algebraic function, it is sufficient to differentiate the equation as it stands. Thus if

$$(1)\qquad x^2+y^2=a^2,$$

we have

$$(2)\qquad D_x x^2+D_x y^2=D_x a^2.$$

To find the value of the second term, apply Theorem V, § 8.

Thus $$D_x y^2=D_y y^2 D_x y=2yD_x y.$$

This last factor, $D_x y$, is precisely the derivative we wish to find, and it is given by completing the differentiations indicated in (2):

$$2x+2yD_x y=0,$$

and solving this equation for $D_x y$:

$$D_x y=-\frac{x}{y}.$$

The final result is, of course, the same as if we had solved equation (1) for y:

$$y=\pm\sqrt{a^2-x^2},$$

and then differentiated:

$$D_xy = \pm\frac{-x}{\sqrt{a^2-x^2}} = -\frac{x}{y}.$$

In the case, however, of the equation

(3) $$x^3 - 2xy + y^5 = 0,$$

we cannot solve for y and obtain an explicit function expressed in terms of radicals. Nevertheless, the equation defines y as a perfectly definite function of x; for, on giving to x any special numerical value, as $x = 2$, we have an algebraic equation for y, — here,

$$y^5 - 4y + 8 = 0,$$

and the roots of this equation can be computed to any degree of precision.

To find the derivative of this function, differentiate equation (3) as it stands with respect to x:

(4) $$D_xx^3 - 2D_x(xy) + D_xy^5 = 0.$$

The second term in this last equation can be evaluated by Theorem III of § 7:

$$D_x(xy) = xD_xy + y,$$

where D_xy denotes the derivative we wish to find.

To the evaluation of the third term in (4) Theorem V of § 8 applies:

$$D_xy^5 = 5y^4D_xy.$$

Hence

$$3x^2 - 2xD_xy - 2y + 5y^4D_xy = 0.$$

Solving this equation for D_xy, we have as the final result:

$$D_xy = \frac{2y-3x^2}{5y^4-2x}.$$

Thus, for example, the curve is seen to go through the point (1, 1), and its slope there is

$$(D_xy)_{(1,1)} = -\tfrac{1}{3}.$$

The differentiation of implicit functions as set forth in the above examples is based on the assumptions a) that the given equation defines y as a function of x; b) that this function has a derivative. The proof of these assumptions belongs to a more advanced stage of analysis. In the case, however, of the equations we meet in practice, — for example, such equations as come from a problem in geometry or physics, — the conditions for the existence of a solution and of its derivative are fulfilled, and we shall take it for granted henceforth that this is true of the implicit functions we meet.

Derivative of x^n, n Fractional. We are now in a position to prove the theorem

$$D_x x^n = nx^{n-1}$$

for the case that n is a fraction. Let

$$n = \frac{p}{q},$$

where p, q are whole numbers which are prime to each other. Let

$$y = x^{\frac{p}{q}}.$$

Then

$$y^q = x^p.$$

Differentiating each side of this equation with respect to x, we have:

$$D_x y^q = D_x x^p,$$

and since, by Theorem V, § 8,

$$D_x y^q = D_y y^q D_x y = qy^{q-1} D_x y,$$

it follows that

$$qy^{q-1} D_x y = px^{p-1},$$

or

$$D_x y = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}}.$$

This last denominator has the value

$$(x^{\frac{p}{q}})^{q-1} = x^{p-\frac{p}{q}}.$$

Hence
$$\frac{x^{p-1}}{y^{q-1}} = \frac{x^{p-1}}{x^{p-\frac{p}{q}}} = x^{\frac{p}{q}-1}.$$

We see, then, that
$$D_x y = \frac{p}{q} x^{\frac{p}{q}-1} = nx^{n-1}, \qquad \text{q. e. d.}$$

If, finally, n is a negative fraction, $n = -m$, the proof can be given precisely as was done in § 7, Ex. 2. Thus the theorem
$$D_x x^n = nx^{n-1}$$
is now established for all commensurable values of n.

The theorem is true even when n is irrational, *e.g.* $n = \pi$ or $\sqrt{2}$; the proof depends on the logarithmic function and will be given when that function has been differentiated.

Example. Differentiate the function
$$y = \sqrt[3]{a^3 - x^3}.$$

Apply Theorem V, § 8, setting
$$z = a^3 - x^3.$$

Then
$$D_x y = D_x z^{\frac{1}{3}} = D_z z^{\frac{1}{3}} D_x z = \tfrac{1}{3} z^{-\frac{2}{3}}(-3x^2).$$

Hence
$$D_x \sqrt[3]{a^3 - x^3} = \frac{-x^2}{\sqrt[3]{(a^3 - x^3)^2}}.$$

EXERCISES

1. If
$$2x^3 - 3x^2y + 4xy + 6y^3 = 0,$$
find $D_x y$.
Ans. $D_x y = \dfrac{6x^2 - 6xy + 4y}{3x^2 - 4x - 18y^2}$.

2. If
$$y^4 - 2xy^2 = x^4,$$
find $D_x y$.

3. Show that the curve
$$x^4 - 2xy^2 + y^3 + 3x - 3y = 0$$
cuts the axis of x at the origin at an angle of 45°.

4. Plot the curve $\quad x^4 + y^4 = 81,$

taking 1 cm. as the unit. Show that this curve is cut orthogonally by the bisectors of the angles made by the coordinate axes.

Differentiate the following functions:

5. $u = \sqrt[5]{1-x}.$ $\qquad$ *Ans.* $\dfrac{-1}{5\sqrt[5]{(1-x)^4}}.$

6. $u = \sqrt[3]{a^2 - 2\,ax + x^2}.$ $\qquad$ *Ans.* $\dfrac{-2}{3\sqrt[3]{a-x}}.$

7. $u = \sqrt[5]{c^3 - 3\,c^2x + 3\,cx^2 - x^3}.$ $\qquad$ *Ans.* $-\dfrac{3}{5\sqrt[5]{c^2 - 2\,cx + x^2}}.$

8. $u = \sqrt[3]{\dfrac{x}{1-x}}.$

9. $u = x\sqrt[4]{a + bx + cx^2}.$

10. $u = \dfrac{\sqrt[4]{a^4 + x^4}}{x}.$ $\qquad$ *Ans. to Ex.* 8. $\dfrac{1}{3\sqrt[3]{x^2(1-x)^4}}.$

11. $\dfrac{\sqrt{a-x}+\sqrt{a+x}}{\sqrt{a-x}-\sqrt{a+x}}.$ $\qquad$ *Ans.* $\dfrac{a^2 + a\sqrt{a^2-x^2}}{x^2\sqrt{a^2-x^2}}.$

12. $y = \sqrt[3]{ax^2}.$

13. $r = \sqrt{a\theta}.$

14. $u = \dfrac{1 - x^{-\frac{1}{2}}}{x^{\frac{1}{5}}}.$ $\qquad$ *Ans.* $D_x u = \dfrac{7 - 2\sqrt{x}}{10\sqrt[10]{x^{17}}}.$

15. $y = \dfrac{1 + x^2}{\sqrt[3]{x}}.$

16. $u = x\sqrt{2\,x}.$

17. $v = 4\sqrt[3]{t^2} + \dfrac{3}{\sqrt{t}} - 1.$

18. $(y^2 + 1)\sqrt{y^3 - y}.$ $\qquad$ *Ans.* $\dfrac{7\,y^4 - 2\,y^2 - 1}{2\sqrt{y^3 - y}}.$

19. $\dfrac{(s^2 - a^2)^{\frac{3}{2}}}{s^3}.$ $\qquad$ *Ans.* $\dfrac{3\,a^2\sqrt{s^2 - a^2}}{s^4}.$

20. $\dfrac{a - x}{\sqrt{2\,ax - x^2}}$.

21. $\sqrt[3]{\dfrac{1 - x}{(1 + x)^2}}$.

22. $v = x(a^2 - x^2)^{\frac{3}{2}}$.

23. $u = (b - t)\sqrt{b + t}$.

24. Find the slope of the curve $y = x^{\frac{1}{5}}$ in the point whose abscissa is 2. *Ans.* $\tan\tau = .115$.

25. If $pv^{1.4} = c$, find $D_v p$.

26. If $y\sqrt{x} = 1 + x$, find $D_x y$. *Ans.* $\dfrac{x - 1}{2\,x\sqrt{x}}$.

27. Differentiate y in two ways, where

$$xy + 4\,y = 3\,x,$$

and show that the results agree.

28. The same, when $y^2 = 2\,mx$.

29. Show that the curves

$$3y = 2x + x^4y^3, \qquad 2y + 3x + y^5 = x^3y,$$

intersect at right angles at the origin.

30. Find the angle at which the curves

$$2\,x = x^4 - xy + x^5, \qquad x^4 + y^4 + 5x = 7\,y,$$

intersect at the origin. *Ans.* $\tan\phi = 1.4$

CHAPTER III

APPLICATIONS

1. Tangents and Normals. By the tangent line, or simply the *tangent*, to a curve at any one of its points, P, is meant the straight line through P whose slope is the same as that of the curve at that point.

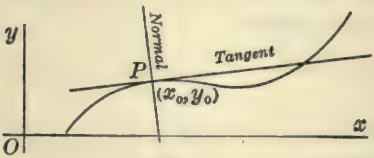


FIG. 9

Let the coordinates of P be denoted by (x_0, y_0). Now, the equation of the straight line through P, whose slope is λ, is

$$y - y_0 = \lambda(x - x_0).$$

On the other hand, the slope of the curve at any point is $D_x y$. If we denote the value of this slope at (x_0, y_0) by $(D_x y)_0$, this will be the desired value of λ:

$$\lambda = (D_x y)_0.$$

Hence the equation of the tangent to the curve

$$y = f(x) \qquad \text{or} \qquad F(x, y) = 0$$

at the point (x_0, y_0) is

$$(1) \qquad y - y_0 = (D_x y)_0 (x - x_0).$$

Since the *normal* is perpendicular to the tangent, its slope, λ', is the negative reciprocal of the slope of that line, or

$$\lambda' = -\frac{1}{(D_x y)_0}.$$

Hence the equation of the normal to the curve at (x_0, y_0) is

$$(2) \quad y - y_0 = -\frac{1}{(D_x y)_0}(x - x_0) \quad \text{or} \quad x - x_0 + (D_x y)_0 \cdot (y - y_0) = 0.$$

Example 1. To find the equation of the tangent to the curve

$$y = x^3$$

in the point $x = \frac{1}{2}$, $y = \frac{1}{8}$. Here

$$D_x y = 3x^2, \qquad (D_x y)_0 = [3x^2]_{x=\frac{1}{2}} = \tfrac{3}{4}.$$

Hence the equation of the tangent is

$$y - \tfrac{1}{8} = \tfrac{3}{4}(x - \tfrac{1}{2}) \qquad \text{or} \qquad 3x - 4y - 1 = 0.$$

Example 2. Let the curve be an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Differentiating the equation as it stands, we get:

$$\frac{2x}{a^2} + \frac{2y}{b^2} D_x y = 0, \qquad D_x y = -\frac{b^2 x}{a^2 y}.$$

Hence the equation of the tangent is

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0).$$

This can be transformed as follows:

$$a^2 y_0 y - a^2 y_0^2 = -b^2 x_0 x + b^2 x_0^2,$$

$$b^2 x_0 x + a^2 y_0 y = a^2 y_0^2 + b^2 x_0^2 = a^2 b^2,$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

EXERCISES

1. Find the equation of the tangent of the curve

$$y = x^3 - x$$

at the origin; at the point where it crosses the positive axis of x. *Ans.* $x + y = 0$; $2x - y - 2 = 0$.

2. Find the equation of the tangent and the normal of the circle

$$x^2 + y^2 = 4$$

at the point $(1, \sqrt{3})$ and check your answer.

3. Show that the equation of the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1.$$

4. Show that the equation of the tangent of the parabola

$$y^2 = 2mx$$

at the point (x_0, y_0) is

$$y_0 y = m(x + x_0).$$

5. Show that the equation of the tangent of the parabola

$$y^2 = m^2 - 2mx$$

at the point (x_0, y_0) is

$$y_0 y = m^2 - m(x + x_0).$$

6. Show that the equation of the tangent of the equilateral hyperbola

$$xy = a^2$$

at the point (x_0, y_0) is

$$y_0 x + x_0 y = 2a^2.$$

7. Find the equation of the tangent to the curve

$$x^3 + y^3 = a^2(x - y)$$

at the origin. *Ans.* $x = y$.

8. Show that the area of the triangle formed by the coordinate axes and the tangent of the hyperbola

$$xy = a^2$$

at any point is constant.

9. Find the equation of the tangent and the normal of the curve

$$x^5 = a^3y^2$$

in the point distinct from the origin in which it is cut by the bisector of the positive coordinate axes.

10. Show that the portion of the tangent of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

at any point, intercepted between the coordinate axes, is constant.

11. The parabola $y^2 = 2ax$ cuts the curve

$$x^3 - 3axy + y^3 = 0$$

at the origin and at one other point. Write down the equation of the tangent of each curve in the latter point.

12. Show that the curves of the preceding question intersect in the second point at an angle of 32° 12′.

2. Maxima and Minima. *Problem.* From a piece of tin 3 ft. square a box is to be made by cutting out equal squares from the four corners and bending up the sides. Determine the dimensions of the box of this description which will hold the most.

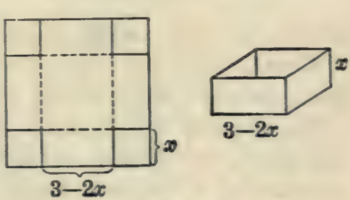


Fig. 10

Solution. Let x be the length of the side of the square removed; then the dimensions of the box are as indicated in the diagrams. Denoting the cubical content of the box by u, we have:

1) $$u = x(3-2x)^2,$$

or

2) $$u = 9x - 12x^2 + 4x^3.$$

The problem is, then, to find the value of x which makes u as large as possible, x being restricted from the nature of the case to being positive and less than $\frac{3}{2}$:

3) $$0 < x < \tfrac{3}{2}.$$

The problem can be treated graphically by plotting the curve 1). We wish to find the highest point on this curve. It appears to be the point for which $x = \frac{1}{2}$, $u = 2$, since other values of x which have been tried lead to smaller values of u.

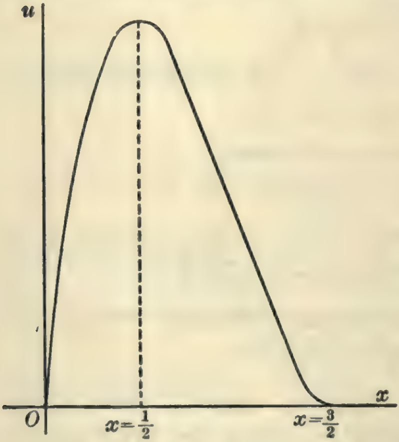


FIG. 11

The foregoing method has the advantage that it is direct, for it assumes no knowledge of mathematics beyond curve plotting. It has the disadvantage that curve plotting, even in the simplest cases, is laborious; and, furthermore, we have not really proved that $x = \frac{1}{2}$ is the best value. We have merely failed to find a better one.

The Calculus supplies a means of meeting both the difficulties mentioned, and yielding a solution with the greatest ease.

The problem is to find the *highest* point on the curve. At this point, the tangent of the curve is evidently parallel to the axis of x. Consequently, the slope of the tangent, *i.e.* $\tan\tau = D_x u$, must have the value 0 here:

$$D_x u = 0.$$

All we need do, therefore, is to compute $D_x u$, most conveniently from equation 2), and set the result equal to 0:

$$D_x u = 9 - 24x + 12x^2 = 0.$$

On solving this quadratic equation for x, we find two roots,

$$x = \tfrac{1}{2}, \quad \tfrac{3}{2}.$$

Only one of these, however, lies within the range 3) of possible values for x, namely, the value $x = \frac{1}{2}$, and hence this is the required value.

EXERCISES

1. Work the foregoing problem for the case that the tin is a rectangle 1 by 2 ft.

Plot accurately the graph, taking 10 cm. as the unit, and determine in this way what appears to be the best value for x, correct to one eighth of an inch.

Solve the problem by the Calculus, and show that the best value for x is .21132 ft., or 2.5359 in.

2. A farmer wishes to fence off a rectangular pasture along a straight river, one side of the pasture being formed by the river and requiring no fence. He has barbed wire enough to build a fence 1000 ft. long. What is the area of the largest pasture of the above description which he can fence off?

3. Show that, of all rectangles having a given perimeter, the square has the largest area.

4. Show that, of all rectangles having a given area, the square has the least perimeter.

5. Each side of a shelter tent is a rectangle 6×8 ft. How must the tent be pitched so as to afford the largest amount of room inside? The ends are to be open.

FIG. 12

Ans. The angle along the ridge-pole must be a right angle.

6. Divide the number 12 into two parts such that the sum of their squares may be as small as possible.

(What is meant is such a division as this: one part might be 4, and then the other would be 8. The sum of the squares would here be $16 + 64 = 80$.)

7. Divide the number 8 into two such parts that the sum of the cube of one part and twice the cube of the other may be as small as possible.

8. Divide the number 9 into two such parts that the product of one part by the square of the other may be as large as possible.

9. Divide the number 8 into two such parts that the product of one part by the cube of the other may be as large as possible.

10. At noon, one ship, which is steaming east at the rate of 20 miles an hour, is due south of a second ship steaming south at 16 miles an hour, the distance between them being 82 miles. If both ships hold their courses, show that they will be nearest to each other at 2 P.M.

11. If, in the preceding problem, the second ship lies to from noon till one o'clock, and then proceeds on her southerly course at 16 miles an hour, when will the ships be nearest to each other?

12. Find the least value of the function

$$y = x^2 + 6x + 10.$$ *Ans.* 1

13. What is the greatest value of the function

$$y = 3x - x^3$$

for positive values of x?

14. For what value of x does the function

$$\frac{12\sqrt{x}}{1 + 4x}$$

attain its greatest value? *Ans.* $x = \frac{1}{4}$

15. At what point of the interval $a < x < b$, a being positive, does the function

$$\frac{x}{(x-a)(b-x)}$$

attain its least value? *Ans.* $x = \sqrt{ab}$.

16. Find the most advantageous length for a lever, by means of which to raise a weight of 490 lb. (see Fig. 13), if the distance of the weight from the fulcrum is 1 ft. and the lever weighs 5 lb. to the foot.

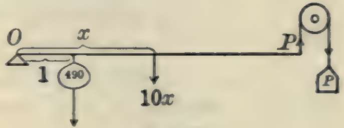


FIG. 13

3. Continuation: Auxiliary Variables. It frequently, — in fact, usually, — happens that it is more convenient to formulate a problem if more variables are introduced at the outset than are ultimately needed. The following examples will serve to illustrate the method.

Example 1. Let it be required to find the rectangle of greatest area which can be inscribed in a given circle.

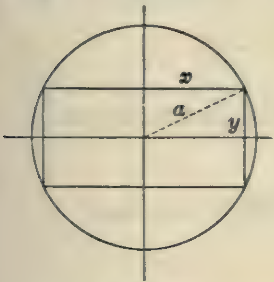


FIG. 14

It is evident that the area of the rectangle will be small when its altitude is small and also when its base is short. Hence the area will be largest for some intermediate shape.

Let u denote the area of the rectangle. Then

(1) $$u = 4\,xy.$$

But x and y cannot both be chosen arbitrarily, for then the rectangle will not in general be inscriptible in the given circle. In fact, it is clear from the Pythagorean Theorem that x and y must satisfy the relation:

(2) $$x^2 + y^2 = a^2.$$

We could now eliminate y between equations (1) and (2), thus obtaining u in terms of x alone; and it is, indeed, im

portant to think of this elimination as performed, for there is only *one* independent variable in the problem. The graph of u, regarded as a function of x, starts at the origin, rises as x increases, but finally comes back to the axis of x again when $x = a$. All this we read off, either from the meaning of u and x in the problem or from equations (1) and (2).

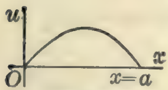


FIG. 15

It is better, however, in practice not to eliminate y, but to differentiate equations (1) and (2) with respect to x as they stand, and then set $D_x u = 0$. Thus from (1),

$$D_x u = 4(y + xD_x y) = 0,$$

and from (2), $$2x + 2yD_x y = 0.$$

From the second of these equations we see that

$$D_x y = -\frac{x}{y}.$$

Substituting for $D_x y$ this value in the first equation, we get:

$$y - \frac{x^2}{y} = 0 \qquad \text{or} \qquad y^2 = x^2.$$

Since x and y are both positive numbers, it follows that

$$y = x.$$

Hence the maximum rectangle is a square.

EXERCISES

1. Work the same problem for an ellipse, instead of a circle.

2. Work the problem for the case of a variable rectangle inscribed in a fixed equilateral triangle.

Example 2. To find the most economical dimensions for a tin dipper. to hold a pint.

Here, the amount of tin required is to be as small as possible, the content of the dipper being given. Let u denote the surface, measured in square inches. Then

a) $$u = 2\pi rh + \pi r^2.$$

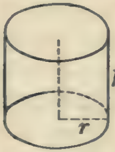


FIG. 16

But r and h cannot both be chosen arbitrarily, for then the dipper would not in general hold a pint. If V denotes the given volume, measured in cubic inches, then, since this volume can also be expressed as $\pi r^2 h$, we have

b) $$\pi r^2 h = V.$$

Differentiating equation a) with respect to r and setting $D_r u = 0$, we have:

$$D_r u = \pi\{2h + 2rD_r h + 2r\} = 0,$$

or

c) $$h + rD_r h + r = 0.$$

Differentiating b) we get:

d) $$\pi\{2rh + r^2 D_r h\} = 0.$$

Now, r cannot $= 0$ in this problem, and so we may divide this last equation through by r, as well as by π:

e) $$2h + rD_r h = 0.$$

It remains to eliminate $D_r h$ between equations c) and e). From e),

$$D_r h = -\frac{2h}{r}.$$

Substituting this value of $D_r h$ in c), we find:

f) $$h - 2h + r = 0, \qquad \text{or} \qquad r = h.$$

Hence the depth of the dipper must just equal its radius.

Discussion. Just what have we done here? The steps we have taken are suggested clearly enough by the solution of

Example 1. We have chosen one of the two variables, r and h, as the independent variable (here, r); differentiated the function u, which is to be made a minimum, with respect to r, and set $D_r u = 0$. Then we differentiated the second equation b), likewise with respect to r, eliminated $D_r u$, and solved. But what does it all mean? What is behind it all?

Just this: the quantity u, in the nature of the case, is a function of r. For, when to r is given *any* positive value, a dipper can be constructed which will fulfill the requirements. Now, if r is very large, we shall have a shallow pan, and evidently the amount of tin required to make it will be large; — *i.e.* u will also have a large value.

But what if r is small? We shall then have a high cylinder of minute cross sections, *i.e.* a pipe. Is it clear that u, the surface, will be large in this case, too? I fear not, for it is purely a relative question as to how high such a pipe must be to hold a pint, and I see no way of guessing intelligently By means of equation b), however, we see that

$$h = \frac{V}{\pi r^2},$$

and if we substitute this value in a), we get

$$u = 2\pi r \frac{V}{\pi r^2} + \pi r^2 = \frac{2V}{r} + \pi r^2.$$

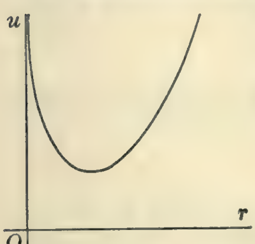


Fig. 17

From this last formula it is clear that, when r is small, u actually is large.

The graph of u, regarded as a function of r, is therefore in character as shown by the accompanying figure. It is a continuous curve lying above the axis of r, very high when r is small, and also very high when r is large. It has, therefore, a lowest point, and for this value of r, the area u of the dipper will be least. But at this lowest point the slope of the curve, $D_r u$, has the value 0. Thus we see, first, that we have a genu-

the minimum problem; — there *is* actually a dipper of smallest area. Secondly, equations c) and d) must hold, and since from these equations it follows by elimination that $r = h$, there is *only one* such dipper, and its radius is equal to its altitude The problem is, then, completely solved.

We inquired merely for the *shape* of the dipper. If the *size* had been asked for, too, it could be found by solving b) and f) for r and h, and expressing V in cubic inches:

$$V = \tfrac{231}{8} = 28.875,$$

$$\pi r^3 = 28.87, \qquad r = 2.095.$$

It can happen in practice that a function attains its greatest or its least value at the end of the interval. In that case, the derivative does not have to vanish. Usually, the facts are patent, and so no special investigation is needed. But it is necessary to assure oneself that a given problem which looks like one of the above does not come under this head, and this is done, as in the cases discussed in the text, by showing that near the ends of the interval the values of the function are larger, for a minimum problem, than for values well within the interval.

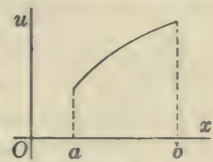


FIG. 18

EXERCISE

Discuss in a similar manner the best shape for a tomato can which is to hold a quart. Here, the tin for the top must also be figured in. Show that the height of such a can should be equal to the diameter of the base. As to the size of the can, its height should be 4.19 in.

A General Remark. It might seem as if the method used in the solution of the above problems were likely to be insecure, since the graph of such a function u might, in the very next problem, look like the accompanying figure. In such a case, there would be several values of x, for each of which $D_x u = 0$,

and we should not know which one to take. Curiously enough, this case does not arise in practice, — at least, I have never come across a physical problem which led to this difficulty. In problems like the above, there must be *at least* one x for which $D_x u = 0$; and when we solve a given problem, we actually find *only one* x which fulfills the condition. Thus there is no ambiguity.

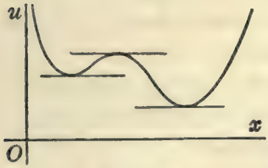


FIG. 19

EXERCISES

1. A 300-gallon tank is to be built with a square base and vertical sides, and is to be lined with copper. Find the most economical proportions.

Ans. The length and breadth must each be double the height.

2. Find the cylinder of greatest volume which can be inscribed in a given cone of revolution.

Ans. Its altitude is one-third that of the cone.

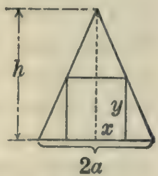


FIG. 20

3. What is the cylinder of greatest convex surface that can be inscribed in the same cone?

Ans. Its altitude is half that of the cone.

4. Of all the cones which can be inscribed in a given sphere, find the one whose lateral area is greatest.

Ans. Its altitude exceeds the radius of the sphere by $33\frac{1}{3}\%$ of that radius.

5. Find the volume of the greatest cone of revolution which can be inscribed in a given sphere.

6. If the top and bottom of the tomato can considered in the Exercise of the text are cut from sheets of tin so that a regular hexagon is used up each time, the waste being a total loss, what will then be the most economical proportions for the can?

7. If the strength of a beam is proportional to its breadth and to the square of its depth, find the shape of the strongest beam that can be cut from a circular log.

Ans. The ratio of depth to breadth is $\sqrt{2}$.

8. Assuming that the stiffness of a beam is proportional to its breadth and to the cube of its depth, find the dimensions of the stiffest beam that can be sawed from a log one foot in diameter.

9. What is the shortest distance from the point (10, 0) to the parabola

$$y^2 = 4x?$$

10. What points of the curve

$$y^2 = x^3$$

are nearest (4, 0)?

11. A trough is to be made of a long rectangular-shaped piece of copper by bending up the edges so as to give a rectangular cross-section. How deep should it be made, in order that its carrying capacity may be as great as possible?

12. Assuming the density of water to be given from 0° to 30° C. by the formula

$$\rho = \rho_0(1 + \alpha t + \beta t^2 + \gamma t^3),$$

where ρ_0 denotes the density at freezing, t the temperature, and

$$\alpha = 5.30 \times 10^{-5}, \qquad \beta = -6.53 \times 10^{-6}, \qquad \gamma = 1.4 \times 10^{-8},$$

show that the maximum density occurs at $t = 4.08°$.

13. Tangents are drawn to the arc of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which lies in the first quadrant. Which one of them cuts off from that quadrant the triangle of smallest area?

14. Work the same problem for the parabola

$$y^2 = a^2 - 4ax.$$

15. Show that, of all circular sectors having the same perimeter, that one has the largest area for which the sum of the two straight sides is equal to the curved side.

4. Increasing and Decreasing Functions. The Calculus affords a simple means of determining whether a function is increasing or decreasing as the independent variable increases. Since the slope of the graph is given by $D_x y$, we see that when $D_x y$ is positive, y increases as x increases, but when $D_x y$ is negative, y decreases as x increases. Figure 21 shows the graph in general when $D_x y$ is positive.

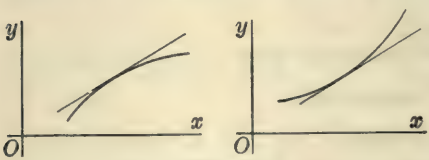


FIG. 21

In each figure both x and y have been taken as positive. But what is said above in the text is equally true when one or both of these variables are negative; for the words *increase* and *decrease* as here used mean *algebraic*, not numerical, increase or decrease. Thus if the temperature is ten degrees below zero (*i.e.* $-10°$) and it changes to eight below ($-8°$), we say the temperature has risen. If we measure the time t, in hours from noon, then 10 A.M. will correspond to $t = -2$. Let u denote the temperature, measured in degrees. Then a temperature chart for 24 hours from midnight to midnight might look like the accompanying figure. At any instant, $t = t'$, for which the slope of the curve, $D_t u$, is positive, the temperature is rising, no matter whether the thermometer is above zero or below, and no matter whether t is positive or negative; and similarly, when $D_t u$ is negative, the temperature is falling.

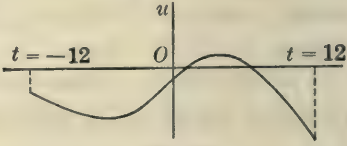


FIG. 22

Again, suppose the amount of business a department store

does in a year, as represented by the net receipts each day, be plotted as a curve ($y =$ receipts, measured in dollars; $x =$ time, measured in days), the curve being smoothed in the usual way. Then a point of the curve at which the derivative is positive (*i.e.* $D_x y > 0$) indicates that, at that time, the business of the firm was increasing; whereas a point at which $D_x y < 0$ means that the business was falling off.

We can state the result in the form of a general theorem, the proof of which is given by inspection of the figure (Fig. 21) and the other forms of the figure, brought out in the above discussion.

THEOREM: *When x increases, then*

(*a*) *if* $D_x y > 0$, *y increases;*

(*b*) *if* $D_x y < 0$, *y decreases.*

Application. As an application consider the condition that a curve $y = f(x)$ have its concave side turned upward, as in Fig. 23. The slope of the curve is a function of x:

$$\tan\tau = \phi(x).$$

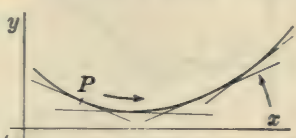


FIG. 23

For, when x is given, a point of the curve, and hence also the slope of the curve at this point, is determined. Consider the tangent line at a variable point P. If we think of P as tracing out the curve and carrying the tangent along with it, the tangent will turn in the counter clock-wise sense, the slope thus increasing algebraically as x increases, whenever the curve is concave upward. And conversely, if the slope increases as x increases, the tangent will turn in the counter clock-wise sense and the curve will be concave upward. Now by the above theorem, when

$$D_x \tan\tau > 0,$$

$\tan\tau$ increases as x increases. Hence the curve is concave upward, when $D_x \tan\tau$ is positive; and conversely.

The derivative $D_x \tan\tau$ is the derivative of the derivative of y. This is called the *second derivative of* y, and is denoted as follows:

$$D_x(D_xy) = D_x^2y$$

(read: "$D\ x$ second of y").*

The test for the curve's being concave downward is obtained in a similar manner, and thus we are led to the following important theorem.

Test for a Curve's being Concave Upward, etc. *The curve*

$$y = f(x)$$

is concave upward *when* $D_x^2y > 0$;

concave downward *when* $D_x^2y < 0$.

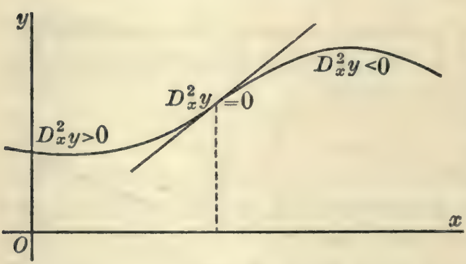


Fig. 24

A point at which the curve changes from being concave upward and becomes concave downward (or *vice versa*) is called a *point of inflection.* Since D_x^2y changes sign at such a point, this function will necessarily, if continuous, vanish there. Hence:

A necessary condition for a point of inflection is that

$$D_x^2y = 0.$$

Example. Consider the curve

$$y = x^3 - 3x.$$

* The derivative of the second derivative, $D_x(D_x^2y)$, is called the *third derivative* and is written D_x^3y, and so on.

Its slope at any point is given by the equation

$$D_x y = 3x^2 - 3.$$

The second derivative of y with respect to x has the value

$$D_x^2 y = 6x.$$

Thus we see that this curve is concave upward for all positive values of x, and concave downward for all negative values. In character it is as shown in the accompanying figure.

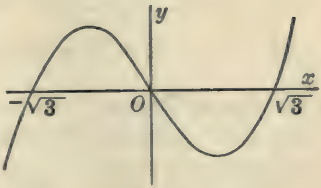


FIG. 25

EXERCISES

For what values of x are the following functions increasing? For what values decreasing?

1. $$y = 4 - 2x^2.$$

2. $$y = x^2 - 2x + 3.$$

Ans. Increasing, when $x > 1$; decreasing, when $x < 1$.

3. $$y = 5 + 12x - x^2.$$

4. $$y = x^3 - 27x + 7.$$

Ans. Increasing, when $x > 3$, and when $x < -3$; decreasing, when $-3 < x < 3$.

5. $y = 5 + 6x - x^3$. **6.** $y = x - x^5$.

7. $$y = x^3 - 9x^2 + 12x - 1.$$

In what intervals are the following curves concave upward; in what, downward?

8. $$y = x^3 - 3x^2 + 7x - 5.$$

Ans. Concave upward, when $x > 1$; concave downward, when $x < 1$.

9. $y = 15 + 8x + 3x^2 - x^3$.

10. $y = x^3 - 6x^2 - x - 1$.

11. $y = 3 - 9x + 24x^2 - 4x^3$.

12. $y = 2x^3 - x^4$.

13. $y = x^4 - 4x^3 - 6x + 11$.

14. $y = -121x + 7x^3 - x^7$.

15. $y = 13 + 23x - 24x^2 + 12x^3 - x^4$.

5. Curve Tracing. In the early work of plotting curves from their equations the only way we had of finding out what the graph of a function looked like was by computing a large number of its points. We are now in possession of powerful methods for determining the character of the graph with scarcely any computation. For, first, we can find the slope of the curve at any point; and, secondly, we can determine in what intervals the curve is concave upward, in what concave downward.*

Example. Let it be required to plot the curve

$$3y = x^3 - 3x^2 + 1. \tag{1}$$

a) Determine first its slope at any point:

$$3D_xy = 3x^2 - 6x, \qquad D_xy = x^2 - 2x. \tag{2}$$

* There are two great applications of the graphical representation of a function. One is *quantitative*, the other, *qualitative*. By the first I mean the use of the graph as a *table*, for actual computation. Thus in the use of logarithms it is desirable to have a graph of the function $y = \log_{10} x$ drawn accurately for values of x between 1 and 10; for by means of such a graph the student can read off the logarithms he is using, correct to two or three significant figures, and so obtain a check on his numerical work.

There is, however, a second large and important class of problems, in which the *character* of a function is the important thing, a minute determination of its values being in general irrelevant.

A case in point is the determination of the number of roots of an algebraic equation, *e.g.* $x^3 - x^2 - 4x + 1 = 0$,

Here, we plot the curve $y = x^3 - x^2 - 4x + 1$

and inquire where it cuts the axis of x. For this purpose it is altogether adequate to know the character of the curve, and for treating this problem the methods of the present paragraph yield a powerful instrument.

It is always useful to know the points at which the tangent to the curve is parallel to the axis of x. These are obtained by setting $D_x y = 0$ and solving. Thus we get from (2) the equation:

$$x^2 - 2x = 0.$$

The roots of this equation are

$$x = 0 \qquad \text{and} \qquad x = 2$$

Now determine accurately the points having these abscissas, plot them, and draw the tangents there:

$$y\,\big|_{x=0} = \tfrac{1}{3}; \qquad y\,\big|_{x=2} = -1.$$

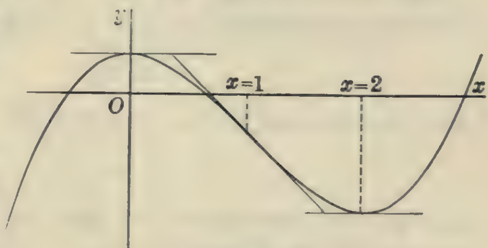


Fig. 26

We do not yet know whether the curve lies above its tangent in one of these points, or below its tangent; it might even cross its tangent, for the point might be a point of inflection. These questions will all be answered by aid of the second derivative.

b) Compute the second derivative:

$$D_x^2 y = 2x - 2 = 2(x - 1).$$

We see that it is positive when x is greater than 1 and negative when x is less than 1:

$$D_x^2 y > 0 \quad \text{when} \quad 1 < x;$$

$$D_x^2 y < 0 \quad \text{when} \quad x < 1.$$

$$D_x^2 y = 0 \quad \text{when} \quad x = 1.$$

Hence the curve has a point of inflection when $x = 1$. This is a most important point on the curve. We will compute its

coordinates accurately, determine the slope of the curve there, and draw accurately the tangent there.

$$y\,|_{x=1} = -\tfrac{1}{3}; \qquad D_x y\,|_{x=1} = -1.$$

This is the last of the important tangents which we need to draw. Since the curve is concave upward to the right of the line $x = 1$, and concave downward to the left of that line, it must be in character as indicated. We see, then, that it cuts the axis of x between 0 and 1, and again to the right of the point $x = 1$; and it cuts that axis a third time to the left of the origin.

These last two points can be located more accurately by computing the function for a few simple values of x.

$$y\,|_{x=3} = \tfrac{1}{3};$$

hence the curve cuts the axis of x between $x = 2$ and $x = 3$.

$$y\,|_{x=-1} = -1;$$

hence the curve cuts the axis between $x = 0$ and $x = -1$.

Incidentally we have shown that the cubic equation

$$x^3 - 3x^2 + 1 = 0$$

has three real roots, and we have located each between two successive integers.

EXERCISES

Discuss in a similar manner the following curves. In particular:

a) Determine the points at which the tangent is horizontal, if such exist, and draw the tangent at each of these points;

b) Determine the intervals in which the curve is concave upward, and those in which it is concave downward;

c) Determine the points of inflection, if any exist, and draw the tangent in each of these points;

d) Draw in the curve.*

In most cases it is desirable to take 2 cm. as the unit.

1. $$y = x^3 + 3x^2 - 2.$$

2. $$y = x^3 - 3x + 1.$$

3. $$y = x^3 + 3x + 1.$$

4. $$6y = 2x^3 - 3x^2 - 12x + 6.$$

5. $$6y = 2x^3 + 3x^2 - 12x - 4.$$

6. $$y = x^3 + x^2 + x + 1.$$

Suggestion. Show that the derivative has no real roots and hence, being continuous, never changes sign.

7. $$12y = 4x^3 - 6x^2 + 12x - 9.$$

8. $$y = 2x^2 - x - x^3.$$

9. $$12y = 4x^3 + 18x^2 + 27x + 12.$$

10. $$y = 1 - 4x + 6x^2 - 3x^3.$$

11. $$y = 1 + 2x + x^2 - x^3.$$

12. $4y = x^4 - 6x^2 + 8.$ 13. $y = x^4 - 8x^2 + 4.$

14. $y = x - x^5.$ 15. $y = x + x^5.$

16. $y = x^4 + x^2.$ 17. $y = x^4 - x^2.$

18. $$y = 3x^5 + 5x^3 + 15x + 2.$$

19. $$60y = 2x^6 + 15x^4 + 60x^2 - 30.$$

6. Relative Maxima and Minima. Points of Inflection. A function

$$y = f(x) \tag{1}$$

* Since a curve separates very slowly from its tangent near a point of inflection, the material graph of the curve must necessarily coincide with the material graph of the tangent for some little distance.

is said to have a *maximum* at a point $x = x_0$ if its value at x_0 is larger than at any other point in the neighborhood of x_0. But such a maximum need not represent the largest value of the function in the complete interval $a \leqq x \leqq b$, as is shown by Fig. 27, and for this reason it is called a *relative maximum*, in distinction from a maximum maximorum, or an *absolute maximum*.

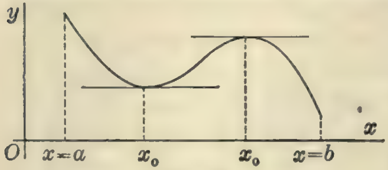


FIG. 27

A similar definition holds for a minimum, the word "larger" merely being replaced by "smaller."

It is obvious that a characteristic feature of a maximum is that the tangent there is parallel to the axis of x, the curve being concave downward. Similarly for a minimum, the curve here being concave upward. Hence the following

TEST FOR A MAXIMUM OR A MINIMUM. *If*

(*a*) $$[D_x y]_{x=x_0} = 0, \qquad [D_x^2 y]_{x=x_0} < 0,$$

the function has a maximum for $x = x_0$; if

(*b*) $$[D_x y]_{x=x_0} = 0, \qquad [D_x^2 y]_{x=x_0} > 0,$$

it has a minimum.

The condition is sufficient, but not necessary; cf. § 7.

Example. Let $$y = x^6 - 3x^2 + 1.$$

Here $$D_x y = 6x^5 - 6x = 6x(x^2 - 1)(x^2 + 1),$$

and hence $$D_x y = 0 \quad \text{for} \quad x = -1, 0, 1.$$

Thus the necessary condition for a maximum or a minimum, $D_x y = 0$, is satisfied at each of the points $x = -1, 0, 1$.

To complete the determination, if possible, compute the second derivative,

$$D_x^2 y = 30x^4 - 6,$$

and determine its sign at each of these points:

$$[D_x^2y]_{x=-1} = 24 > 0, \qquad \therefore x = -1 \text{ gives a } \textit{minimum};$$
$$[D_x^2y]_{x=0} = -6 < 0, \qquad \therefore x = 0 \text{ gives a } \textit{maximum};$$
$$[D_x^2y]_{x=1} = 24 > 0, \qquad \therefore x = 1 \text{ gives a } \textit{minimum}.$$

Points of Inflection. A point of inflection is characterized geometrically by the phenomenon that, as a point P describes the curve, the tangent at P ceases rotating in the one direction and, turning back, begins to rotate in the opposite direction. Hence the slope of the curve, $\tan \tau$, has either a maximum or a minimum at a point of inflection.

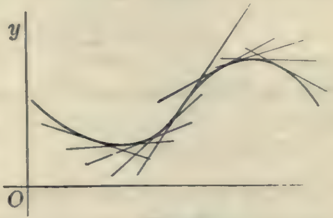


FIG. 28

Conversely, if $\tan \tau$ has a maximum or a minimum, the curve will have a point of inflection. For, suppose $\tan \tau$ is at a maximum when $x = x_0$. Then as x, starting with the value x_0, increases, $\tan \tau$, *i.e.* the slope of the curve, decreases algebraically, and so the curve is concave downward to the right of x_0. On the other hand, as x decreases, $\tan \tau$ also decreases, and so the curve is concave upward to the left of x_0.

Now, we have just obtained a theorem which insures us a maximum or a minimum in the case of any function which satisfies the conditions of the theorem. If, then, we choose as that function, $\tan \tau$, the theorem tells us that $\tan \tau$ will surely be at a maximum or a minimum if

$$D_x \tan \tau = 0, \qquad D_x^2 \tan \tau \neq 0.$$

Hence, remembering that

$$\tan \tau = D_x y,$$

we obtain the following

TEST FOR A POINT OF INFLECTION. *If*

$$[D_x^2y]_{x=x_0} = 0, \qquad [D_x^3y]_{x=x_0} \neq 0,$$

the curve has a point of inflection at $x = x_0$.

This test, like the foregoing for a maximum or a minimum, is sufficient, but not necessary; cf. § 7.

Example. Let

$$27\,y = x^4 + 2x^3 - 12\,x^2 + 14\,x - 1.$$

Then

$$27\,D_xy = 4\,x^3 + 6\,x^2 - 24\,x + 14,$$

$$27\,D_x^2y = 12\,x^2 + 12\,x - 24 = 12(x-1)(x+2),$$

$$27\,D_x^3y = 12(2\,x+1).$$

Setting $D_x^2y = 0$, we get the points $x = 1$ and $x = -2$. And since

$$27[D_x^3y]_{x=1} = 36 \neq 0, \qquad 27[D_x^3y]_{x=-2} = -36 \neq 0,$$

we see that both of these points are points of inflection.

The slope of the curve in these points is given by the equations:

$$27[D_xy]_{x=1} = 0, \qquad 27[D_xy]_{x=-2} = 54.$$

Hence the curve is parallel to the axis of x at the first of these points; at the second its slope is 2.

EXERCISES

Test the following curves for maxima, minima, and points of inflection, and determine the slope of the curve in each point of inflection.

1. $y = 4\,x^3 - 15\,x^2 + 12\,x + 1.$
2. $y = x^3 + x^4 + x^5.$
3. $6\,y = x^6 - 3\,x^4 + 3\,x^2 - 1.$
4. $y = (x-1)^3(x+2)^2.$

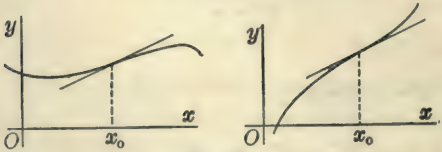


FIG. 29

5. $y = \dfrac{x}{2 + 3\,x^2}.$
6. $y = (1 - x^2)^3.$
7. Deduce a test for distinguishing between two such points of inflection as those indicated in Fig. 29.

7. Necessary and Sufficient Conditions. In order to understand the nature of the tests obtained in the foregoing paragraph it is essential that the student have clearly in mind the meaning of a *necessary condition* and of a *sufficient condition.* Let us illustrate these ideas by means of some simple examples.

a) A *necessary* condition that a quadrilateral be a square is that its angles be right angles. But the condition is obviously not sufficient; all rectangles also satisfy it.

b) A *sufficient* condition that a quadrilateral be a square is that its angles be right angles and each side be 4 in. long. But the condition is obviously not necessary; the sides might be 6 in. long.

c) A *necessary and sufficient* condition that a quadrilateral be a square is that its angles be right angles and its sides be mutually equal.

As a further illustration consider the following. It is a well-known fact about whole numbers that if the sum of the digits of a whole number is divisible by 3, the number is divisible by 3; and conversely. Also, if the sum of the digits of a whole number is divisible by 9, the number is divisible by 9; and conversely. Hence we can say:

i) A *necessary* condition that a whole number be divisible by 9 is that the sum of its digits be divisible by 3. But the condition is not sufficient.

ii) A *sufficient* condition that a whole number be divisible by 3 is that the sum of its digits be divisible by 9. But the condition is not necessary.

iii) A *necessary and sufficient* condition that a whole number be divisible by 3 (or 9) is that the sum of its digits be divisible by 3 (or 9).

Turning now to the considerations of § 6, we see that a *necessary* condition for a minimum is that

$$D_x y = 0$$

at the point in question, $x = x_0$. But this condition is not sufficient. When it is fulfilled, the function may have a maximum, or it may have a point of inflection with horizontal tangent.

On the other hand, the condition

$$[D_x y]_{x=x_0} = 0, \qquad [D_x^2 y]_{x=x_0} > 0$$

is *sufficient* for a minimum. But it is not necessary. Thus the function

$$(1) \qquad y = x^4$$

obviously has a minimum when $x = 0$. The necessary condition, $D_x y = 0$, is of course fulfilled:

$$D_x y = 4\,x^3, \qquad [D_x y]_{x=0} = 0.$$

But here $\quad D_x^2 y = 12\,x^2, \quad$ and $\quad [D_x^2 y]_{x=0}$

is not positive; it is 0.

Again, as was shown in § 4, a *necessary* condition for a point of inflection is that

$$D_x^2 y = 0$$

at that point. But this condition is not sufficient. Thus in the case of the curve (1) this condition is fulfilled at the origin. But the origin is not a point of inflection.

Remark. It may seem to the student that such tests are unsatisfactory since they do not apply to all cases and thus appear to be incomplete. But their very strength lies in the fact that they do not tell the truth in too much detail. They single out the big thing in the cases which arise in practice and yield criteria which can be applied with ease to the great majority of these cases.

8. Velocity; Rates. By the *average velocity* with which a point moves for a given length of time t is meant the distance s traversed divided by the time:

$$\text{average velocity} = \frac{s}{t}.$$

Thus a railroad train which covers the distance between two stations 15 miles apart in half an hour has an average speed of $15/\frac{1}{2} = 30$ miles an hour.

When, however, the point in question is moving sometimes fast and sometimes slowly, we can describe its speed approximately at any given instant by considering a short interval of time immediately succeeding the instant t_0 in question, and taking the average velocity for this short interval.

For example, a stone dropped from rest falls according to the law:

$$s = 16t^2.$$

To find how fast it is going after the lapse of t_0 seconds. Here

(1) $$s_0 = 16t_0^2.$$

A little later, at the end of t' seconds from the beginning of the fall,

(2) $$s' = 16t'^2$$

and the average velocity for the interval of $t' - t_0$ seconds is

(3) $$\frac{s' - s_0}{t' - t_0} \text{ ft. per second.}$$

Let us consider this average velocity, in particular, after the lapse of 1 second:

$$t_0 = 1, \qquad s_0 = 16.$$

Let the interval of time, $t' - t_0$, be $\frac{1}{10}$ sec. Then

$$s' = 16 \times 1.1^2 = 19.36,$$

$$\frac{s' - s_0}{t' - t_0} = \frac{3.36}{.1} = 33.6 \text{ ft. a second.}$$

Thus the average velocity for one-tenth of a second immediately succeeding the end of the first second of fall is 33.6 ft. a second.

Next, let the interval of time be $\frac{1}{100}$ sec. Then a similar computation gives, to three significant figures:

$$\frac{s' - s_0}{t' - t_0} = 32.2 \text{ ft. a second.}$$

And when the interval is taken as $\frac{1}{1000}$ sec., the average velocity is 32.0 ft. a second.

These numerical results indicate that we can get at the speed of the stone at any desired instant to any desired degree of accuracy by direct computation; we need only to reckon out the average velocity for a sufficiently short interval of time succeeding the instant in question.

We can proceed in a similar manner when a point moves according to any given law. Can we not, however, by the aid of the Calculus avoid the labor of the computations and at the same time make precise exactly what is meant by *the velocity* of the point *at a given instant?* If we regard the interval of time $t' - t_0$ as an increment of the variable t and write $t' - t_0 = \Delta t$, then $s' - s_0 = \Delta s$ will represent the corresponding increment in the function, and thus we have:

$$\text{average velocity} = \frac{\Delta s}{\Delta t}.$$

Now allow Δt to approach 0 as its limit. Then the average velocity will in general approach a limit, and *this limit we take as the definition of the velocity, v, at the instant t_0*:

$$\lim\,(\text{average velocity from } t = t_0 \text{ to } t = t')$$

$$= \text{actual velocity}^* \text{ at instant } t = t_0,$$

or
$$v = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = D_t s.$$

Hence it appears that the velocity of a point is the *time-derivative* of the space it has traveled. In the case of a freely falling body this velocity is

$$v = D_t s = 32\,t.$$

In the foregoing definition, s has been taken as the distance actually traversed by the moving point, P. More generally, let s denote the length of the arc of the curve on which P is moving, s being measured from an arbitrarily chosen fixed

* Sometimes called the *instantaneous velocity*.

point of that curve. Either direction along the curve may be chosen as the positive sense for s. Thus, in the case of a freely falling body, s might be taken as the distance of the body above the ground. If h denotes the initial distance, then

$$s + s' = h,$$

where s' denotes the distance actually traversed by P at any given instant. Hence

$$D_ts + D_ts' = 0,$$

or

$$D_ts = -D_ts'.$$

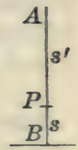


FIG. 30

Here D_ts gives *numerically* the value of the velocity, but D_ts is a negative quantity.

We will, accordingly, extend the conception of velocity, defining the velocity v of the point as D_ts:

$$v = D_ts.$$

Thus the numerical value of v or D_ts will always give the *speed*, or the value of the velocity in the earlier sense. In case s increases with the time, D_ts is positive and represents the speed. If, however, s decreases with the time, D_ts is negative, and the velocity, v, is therefore here negative, the speed now being given by $-v$ or $-D_ts$. In all cases,

$$\text{Speed} = |v| = |D_ts|.$$

Example. Let a body be projected upward with an initial velocity of 96 ft. a second. Assuming from Physics the law that

$$s = 96t - 16t^2,$$

find its velocity a) at the end of 1 sec.

b) at the end of 5 sec.

Solution. By definition, the velocity at any instant is

$$v = D_ts = 96 - 32t.$$

Hence

a) $$v|_{t=1} = 64.$$

b) $$v|_{t=5} = -64.$$

The meaning of these results is that, at each of the two instants, the speed is the same, namely, 64 ft. a second (and the height above the ground is also seen to be the same, $s = 80$ ft.). But when $t = 1$, $D_t s$ is positive; hence s is increasing with the time and the body is rising. When $t = 5$, $D_t s$ is negative; hence s is decreasing with the time, and the body is descending.

Rates. Consider any length or distance, r, which is changing with the time, and so is a function of the time. Let r_0 denote the value of r at a given instant, $t = t_0$, and let r' be the value of r at a later instant, $t = t'$. Then the increase in r will be $r' - r_0 = \Delta r$ and that in t will be $t' - t_0 = \Delta t$. Thus in the interval of time of Δt seconds succeeding the instant $t = t_0$,

$$\text{average rate of increase of } r = \frac{\Delta r}{\Delta t}.$$

Now let Δt approach 0 as its limit. Then the average rate of increase will in general approach a limit, and *this limit we take as the definition of the rate of increase of* r *at the instant* t_0:

$$\lim (\text{average rate of increase from } t = t_0 \text{ to } t = t')$$

$$= \text{actual rate of increase at instant } t = t_0$$

$$= \lim_{\Delta t \doteq 0} \frac{\Delta r}{\Delta t} = D_t r.$$

In other words, the *rate at which* r *is increasing* at any instant is defined as the *time-derivative* of r.

If r is decreasing, $D_t r$ will be a negative quantity; and conversely, if $D_t r$ is negative, then r is decreasing. In either case, the *numerical* value of $D_t r$ gives the *rate of change* of r; just as, in the case of velocities, the numerical value of $D_t s$ gives the speed.

More generally, instead of r, we may have any physical quantity, u, as an area or a volume or the current in an electric circuit or the number of calories in a given body.

In all these cases, the rate at which u is increasing is defined as the *time-derivative* of u, *i.e.* as $D_t u$; and the rate of change of u is $|D_t u|$.

Example. At noon, one ship is steaming east at the rate of 18 miles an hour, and a second ship, 40 miles north of the first, is steaming south at the rate of 20 miles an hour. At what rate are they separating from each other at one o'clock?

Solution. The relation between r and t is here given by the Pythagorean Theorem:

$$r^2 = (40 - 20t)^2 + (18t)^2,$$

or

(1) $$r^2 = 1600 - 1600t + 724t^2.$$

Hence

(2) $$r = \sqrt{1600 - 1600t + 724t^2}.$$

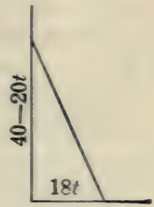


FIG. 31

We wish to find $D_t r$. This can be done by differentiating equation (2); but that would be poor technique, since it is simpler to differentiate equation (1) through with respect to t:

$$2rD_t r = -1600 + 1448t,$$

(3) $$D_t r = \frac{-800 + 724t}{r}.$$

Equation (3) gives the rate at which r is increasing at any instant t; *i.e.* t hours past noon, or at t o'clock.

Setting now, in particular, $t = 1$, we obtain:

$$D_t r\,\big|_{t=1} = -\frac{76}{\sqrt{724}} = -2.825.$$

The meaning of this result is twofold. First, since $D_t r$ is negative when $t = 1$, the ships are not receding from each other, but are coming nearer together. Secondly, the rate of change of the distance between them is, at one o'clock, 2.825 miles an hour.

Let the student determine how long they will continue to approach each other, and what the shortest distance between them will be.

Remark. It is important for the student to reflect on the method of solution of this problem, since it is typical. We were asked to find the rate of recession *at just one instant*, $t=1$. We began by determining the rate of recession generally, *i.e.* for an *arbitrary instant*, $t=t$. Having solved the general problem, we then, as the last step in the process, brought into play the specific value of t which alone we cared for, namely, $t=1$.

The student will meet this method again and again, — in integration, in mechanics, in series, etc. We can formulate the foregoing remark suggestively as follows: By means of the Calculus we can often determine a particular physical quantity, like a velocity, an area, or the time it takes a body, acted on by known forces, to reach a certain position. The method consists in first determining a *function*, whereby the general problem is solved for the variable case; and then, as the last step in the process, the special numerical values with which alone the proposed question is concerned, are brought into play.

EXERCISES

1. The height of a stone thrown vertically upward is given by the formula:

$$s=48t-16t^2.$$

When it has been rising for one second, find (*a*) its average velocity for the next $\frac{1}{10}$ sec.; (*b*) for the next $\frac{1}{100}$ sec.; (*c*) its actual velocity at the end of the first second; (*d*) how high it will rise.

Ans. (*a*) 14.4 ft. a second; (*b*) 15.84 ft. a second; (*c*) 16 ft. a second; (*d*) 36 ft.

2. One ship is 80 miles due south of another ship at noon, and is sailing north at the rate of 10 miles an hour. The

second ship sails west at the rate of 12 miles an hour. Will the ships be approaching each other or receding from each other at 2 o'clock? What will be the rate at which the distance between them is changing at that time? How long will they continue to approach each other?

3. If two ships start abreast half a mile apart and sail due north at the rates of 9 miles an hour and 12 miles an hour, how far apart will they be at the end of half an hour? How fast will they be receding at that time?

4. Two ships are steaming east, one at the rate of 18 miles an hour, the other at the rate of 24 miles an hour. At noon, one is 50 miles south of the other. How fast are they separating at 7 P.M.?

5. A ladder 20 ft. long rests against a house. A man takes hold of the lower end of the ladder and walks off with it at the uniform rate of 2 ft. a second. How fast is the upper end of the ladder coming down the wall when the man is 4 ft. from the house?

6. A kite is 150 ft. high and there are 250 ft. of cord out. If the kite moves horizontally at the rate of 4 m. an hour directly away from the person who is flying it, how fast is the cord being paid out? *Ans.* $3\frac{1}{5}$ m. an hour.

7. A stone is dropped into a placid pond and sends out a series of concentric circular ripples. If the radius of the outer ripple increases steadily at the rate of 6 ft. a second, how rapidly is the area of the water disturbed increasing at the end of 2 sec.? *Ans.* 452 sq. ft. a second.

8. A spherical raindrop is gathering moisture at such a rate that the radius is steadily increasing at the rate of 1 mm. a minute. How fast is the volume of the drop increasing when the diameter is 2 mm.?

9. A man is walking over a bridge at the rate of 4 miles an hour, and a boat passes under the bridge immediately below him rowing 8 miles an hour. The bridge is 20 ft. above the

boat. How fast are the boat and the man separating 3 minutes later?

Suggestion. The student should make a space model for this problem by means, for example, of the edge of a table, a crack in the floor, and a string; or by two edges of the room which do not intersect, and a string. He should then make a drawing of his model such as is here indicated.

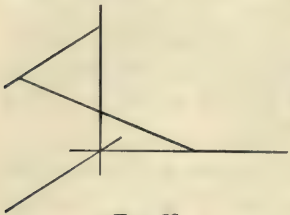


FIG. 32

10. A locomotive running 30 miles an hour over a high bridge dislodges a stone lying near the track. The stone begins to fall just as the locomotive passes the point where it lay. How fast are the stone and the locomotive separating 2 sec. later? *

11. Solve the same problem if the stone drops from a point 40 ft. from the track and at the same level, when the locomotive passes.

12. A lamp-post is distant 10 ft. from a street crossing and 60 ft. from the houses on the opposite side of the street. A man crosses the street, walking on the crossing at the rate of 4 miles an hour. How fast is his shadow moving along the walls of the houses when he is halfway over?

* Bôcher, *Plane Analytic Geometry*, p. 230.

CHAPTER IV

INFINITESIMALS AND DIFFERENTIALS

1. Infinitesimals. An *infinitesimal* is a variable which it is desirable to consider only for values numerically small and which, when the formulation of the problem in hand has progressed to a certain stage, is allowed to approach 0 as its limit.

Thus in the problem of differentiation, or finding the limit

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x y, \tag{1}$$

Δx and Δy are infinitesimals; for we allow Δx to approach 0 as its limit, and then Δy also approaches 0.

Again, if we denote the value of the difference $\Delta y / \Delta x - D_x y$ by ϵ, so that

$$\frac{\Delta y}{\Delta x} - D_x y = \epsilon, \tag{2}$$

then ϵ is an infinitesimal. For, when Δx approaches 0, the left-hand side of equation (2) approaches 0, and so ϵ is a variable which approaches 0 as its limit, *i.e.* an infinitesimal.

Principal Infinitesimal. When we are dealing with a number of infinitesimals, α, β, γ, etc., it is usually possible to choose any one of them as the independent variable, the others then becoming functions of it, or dependent variables. That infinitesimal which is chosen as the independent variable is called the *principal infinitesimal.*

Thus, if the infinitesimals are α and β, and if

$$\beta = \frac{2\alpha}{1 + 3\alpha}, \tag{3}$$

it is natural to choose α as the principal infinitesimal. But it is perfectly possible to take β as the principal infinitesimal. α then becomes the dependent variable, and is expressed in terms of β by solving equation (3) for α:

$$\alpha = \frac{\beta}{2 - 3\beta}. \tag{4}$$

Order of Infinitesimals. We are going to separate infinitesimals into classes, according to the relative speed with which they approach 0. Suppose we let α set the pace, taking on the values .5, .1, .01, .001, etc. Consider, for example, α^2. Then α^2 takes on the respective values .25, .01, .0001, etc., and hence runs far ahead of α:

α	.5	.1	.01	.001 ⋯
α^2	.25	.01	.0001	.000001 ⋯

Furthermore, the closer the two get to 0, the *relatively* nearer α^2 is to 0. Thus, when $\alpha = .5$, α^2 is twice as close; but when $\alpha = .01$, α^2 is one hundred times as close; and so on.

Again, consider the infinitesimal $\frac{1}{2}\alpha$. It is always twice as close to 0 as α is. Similarly, 10α is always one-tenth as close as α.

From these examples we see that there is a decided difference between the *relative* behavior of α and $k\alpha$ on the one hand, and that of α and α^2 on the other. For, $k\alpha$ is keeping pace *relatively* with α, whereas α^2 runs indefinitely ahead of α, *relatively.* Consequently, we should put $k\alpha$ into the same class with α, whereas α^2 forms the starting point for a new class. To this latter class would belong such infinitesimals as $\frac{1}{2}\alpha^2$ or $4\alpha^2 - \alpha^3$; and the former class would include, for example, $2\alpha + 3\alpha^2$ and $\frac{1}{100}\alpha - 1000\alpha^3$. Let the student make out a table like the above for each of these examples.

What is the common property of all infinitesimals of the same class? Is it not, that, for two infinitesimals, the *relative* speed with which they approach 0 is nearly, or quite, a fixed number not zero? It is this idea which lies at the bottom of

the conception of the *order* of an infinitesimal, and it is formulated in a precise definition as follows:

DEFINITION. Two infinitesimals, β and γ, are said to be of the *same order* if their ratio approaches a limit not 0:

$$\lim \frac{\beta}{\gamma} = K \neq 0.$$

Thus $\quad \beta = 2\alpha + \alpha^2 \quad$ and $\quad \gamma = 3\alpha - \alpha^3$

are of the same order. For,

$$\frac{\beta}{\gamma} = \frac{2\alpha + \alpha^2}{3\alpha - \alpha^3} = \frac{2 + \alpha}{3 - \alpha^2},$$

and hence, when α approaches 0,

$$\lim \frac{\beta}{\gamma} = \lim \frac{2 + \alpha}{3 - \alpha^2} = \frac{2}{3} \neq 0.$$

Similarly, $12\alpha^2 + 3\alpha^5$ and $6\alpha^2 - 7\alpha^3$ are infinitesimals of the same order.

An infinitesimal β is said to be of *higher order* than γ if

$$\lim \frac{\beta}{\gamma} = 0,$$

Thus if $\quad \beta = 9\alpha^2 \quad$ and $\quad \gamma = 2\alpha + 5\alpha^4,$

β is of higher order than γ. For,

$$\frac{\beta}{\gamma} = \frac{9\alpha^2}{2\alpha + 5\alpha^4} = \frac{9\alpha}{2 + 5\alpha^3},$$

and hence, when α approaches 0,

$$\lim \frac{\beta}{\gamma} = \lim \frac{9\alpha}{2 + 5\alpha^3} = 0.$$

Finally, β is said to be of *lower order* than γ if

$$\lim \frac{\beta}{\gamma} = \infty, \tag{5}$$

(read: "β/γ *becomes infinite*"; NOT "β/γ equals infinity." *).

* The student should now turn back to Chapter II, § 5, and read again carefully what is said there about infinity. In particular, he should im-

Thus if $\beta = \sqrt{\alpha}$ and $\gamma = 6\alpha + \alpha^3$,

β is of lower order than γ. For

$$\frac{\beta}{\gamma} = \frac{\sqrt{\alpha}}{6\alpha + \alpha^3} = \frac{1}{\sqrt{\alpha}(6 + \alpha^2)}.$$

When α approaches 0, it is evident that the last fraction increases without limit, or

$$\lim \frac{\beta}{\gamma} = \infty.$$

First Order, Second Order, etc. An infinitesimal β is said to be of the *first order* if it is of the same order as the principal infinitesimal, α; *i.e.* if

$$\lim \frac{\beta}{\alpha} = K \neq 0.$$

If β is of the same order as α^2, *i.e.* if

$$\lim \frac{\beta}{\alpha^2} = K \neq 0,$$

then β is said to be of the *second order.* And, generally, if β is of the same order as α^n, *i.e.* if

$$\lim \frac{\beta}{\alpha^n} = K \neq 0,$$

then β is said to be of the *n-th order.*

Thus if

$$\beta = 2\alpha \quad \text{or} \quad \beta = \frac{\alpha}{2 - \alpha} \quad \text{or} \quad \beta = \alpha + \alpha^2,$$

then β is of the first order.

But if

$$\beta = 2\alpha^2 + \alpha^3 \quad \text{or} \quad \beta = \frac{\alpha^2}{3 + \alpha} \quad \text{or} \quad \beta = \alpha^2,$$

then β is of the second order.

press on his mind the fact that infinity is not a limit and that in the notation used in (5) the = sign does *not* mean that one number is equal to another number. The formula is not an equation in the sense in which $2x = 3$ or $a^2 - b^2 = (a - b)(a + b)$ is an equation. The formula means no more and no less than that the variable β/γ increases in value without limit.

If $\beta = \sqrt{a}$, then
$$\frac{\beta}{\alpha^{\frac{1}{2}}} = 1,$$
and
$$\lim \frac{\beta}{\alpha^{\frac{1}{2}}} = 1 \neq 0.$$
Hence β is of the order $\frac{1}{2}$.

It is easily seen that if two infinitesimals β and γ are, under the present definition, each of order n, then they also satisfy the earlier definition of being of the same order. For, let

$$\lim \frac{\beta}{\alpha^n} = K \neq 0 \qquad \text{and} \qquad \lim \frac{\gamma}{\alpha^n} = L \neq 0.$$

Then, if we denote the differences $\beta/\alpha^n - K$ and $\gamma/\alpha^n - L$ respectively by ϵ and η, so that

$$(6) \qquad \frac{\beta}{\alpha^n} - K = \epsilon \qquad \text{and} \qquad \frac{\gamma}{\alpha^n} - L = \eta,$$

these variables, ϵ and η, will be infinitesimals. For, the left-hand side of each of the equations (6) approaches 0.

From equations (6) it follows that

$$\frac{\beta}{\alpha^n} = K + \epsilon \qquad \text{and} \qquad \frac{\gamma}{\alpha^n} = L + \eta.$$

On dividing one of these equations by the other we have:

$$\frac{\beta}{\gamma} = \frac{K + \epsilon}{L + \eta}.$$

We are now ready to allow α to approach 0 as its limit. Then

$$\lim \frac{\beta}{\gamma} = \lim \frac{K + \epsilon}{L + \eta}.$$

By Theorem III of Chapter 2, § 5 this last limit has the value

$$\lim \frac{K + \epsilon}{L + \eta} = \frac{\lim (K + \epsilon)}{\lim (L + \eta)} = \frac{K}{L}.$$

Hence, finally
$$\lim \frac{\beta}{\gamma} = \frac{K}{L} \neq 0, \qquad \text{q. e. d.}$$

EXERCISES

1. Show that

$$\beta = 5\alpha - 11\alpha^2 + \alpha^3 \qquad \text{and} \qquad \gamma = 7\alpha + \alpha^4$$

are infinitesimals of the same order.

2. Show that

$$\beta = 2\alpha - 3\alpha^2 \qquad \text{and} \qquad \gamma = 2\alpha + \alpha^4$$

are infinitesimals of the same order, but that their difference, $\beta - \gamma$, is of higher order than β (or γ).

3. Show that $\beta = \dfrac{7\alpha^2}{\alpha^3 - 2}$ is an infinitesimal of the second order, referred to α as principal infinitesimal.

4. Show that $\beta = \sqrt{\alpha^2 + 2\alpha^5}$ is of the first order, referred to α.

5. Show that $\beta = \sqrt{2\alpha + 13\alpha^3}$ is of lower order than α.

6. Show that the order of β in question 5 is $n = \frac{1}{2}$.

Determine the order of each of the following infinitesimals, referred to α as the principal infinitesimal:

7. $\frac{1}{2}\alpha + 18\alpha^3$.

8. $-\alpha + \sqrt{2\alpha^3 + \alpha^4}$.

9. $\dfrac{7\alpha^2}{13 - \alpha}$.

10. $\sqrt{\dfrac{2\alpha^2 + \alpha^5}{8 - 7\alpha}}$.

11. $\sqrt[3]{\alpha^3 - \alpha}$.

12. $\sqrt[13]{-\alpha^{12} + \alpha^{13}}$.

13. $\sqrt[5]{2\alpha^2 - \alpha^3}$.

14. $\sqrt[4]{\dfrac{3\alpha^6 + 4\alpha^4}{\alpha^2 + 2}}$.

15. If β and γ are infinitesimals of orders n and m respectively, show that their product, $\beta\gamma$, is an infinitesimal of order $n + m$.

16. If β and γ are infinitesimals of the same order, show that their sum is, in general, an infinitesimal of the same order.

Are there exceptions? Illustrate by examples.

2. Continuation; Fundamental Theorem. *Principal Part of an Infinitesimal.* Let β be an infinitesimal of order n, and let α be the principal infinitesimal. Then

$$\lim \frac{\beta}{\alpha^n} = K \neq 0.$$

Moreover, as pointed out in the last paragraph,

$$(1) \qquad \frac{\beta}{\alpha^n} = K + \epsilon,$$

where ϵ is infinitesimal. From (1) it follows that

$$(2) \qquad \beta = K\alpha^n + \epsilon\alpha^n.$$

This last equation gives a most important analysis (*i.e.* breaking up) of β into two parts, each of which is simple for its own peculiar reason.

i) $K\alpha^n$ is the simplest infinitesimal of the nth order imaginable, — a monomial in the independent variable, the function

$$y = Kx^n.$$

ii) $\epsilon\alpha^n$ is an infinitesimal of higher order than the nth.

The first part, $K\alpha^n$, is called the *principal part* of β.

By far the most important case in practice is that of infinitesimals β of the *first* order, $n = 1$. Here

$$\frac{\beta}{\alpha^n} = \frac{\beta}{\alpha} = K + \epsilon$$

and

$$\beta = K\alpha + \epsilon\alpha.$$

Hence we see that *the principal part of an infinitesimal of the first order is proportional to the principal infinitesimal.*

Example 1. Let $\qquad \beta = 2\alpha - \alpha^2.$

Then β is obviously of the first order, or $n = 1$, and here

$$\frac{\beta}{\alpha^n} = \frac{\beta}{\alpha} = 2 - \alpha.$$

Clearly, then, $K = 2, \qquad \epsilon = -\alpha,$

and the principal part of β is 2α.

Example 2. Let $$\beta = \frac{2\alpha^2}{7-4\alpha}.$$

Here, obviously, $n = 2$, and

$$\lim \frac{\beta}{\alpha^2} = \lim \frac{2}{7-4\alpha} = \frac{2}{7}.$$

Hence $K = \frac{2}{7}$. By definition,

$$\epsilon = \frac{\beta}{\alpha^n} - K.$$

In the present case, then,

$$\epsilon = \frac{2}{7-4\alpha} - \frac{2}{7} = \frac{8\alpha}{7(7-4\alpha)}.$$

EXERCISE

Determine the principal parts of a goodly number of the infinitesimals occurring in the Exercises at the end of § 1.

Equivalent Infinitesimals. Two infinitesimals, as β and γ, shall be said to be *equivalent* if the limit of their ratio is unity:

$$\lim \frac{\beta}{\gamma} = 1.$$

For example, the following pairs of infinitesimals are equivalent:

i) $2\alpha + \alpha^2$ and $2\alpha + \alpha^3$;

ii) $\frac{1}{2}\alpha^2 - \alpha^3$ and $\frac{1}{2}\alpha^2 + \alpha^3$;

iii) $\sqrt{2\alpha + 5\alpha^2}$ and $\sqrt{2\alpha - 7\alpha^4}$.

An infinitesimal and its principal part are always equivalent infinitesimals. For, if $K\alpha^n$ is the principal part of β, then

$$\beta = K\alpha^n + \eta,$$

where η is of higher order than $K\alpha^n$. Hence

$$\frac{\beta}{K\alpha^n} = 1 + \frac{\eta}{K\alpha^n}, \qquad \lim \frac{\beta}{K\alpha^n} = 1 + \lim \frac{\eta}{K\alpha^n}.$$

But $\lim \eta/K\alpha^n = 0$, and the statement is established.

Two infinitesimals which have the same principal parts are equivalent, and conversely.

Equivalent infinitesimals are of the same order; but the converse is not true.

The difference between two equivalent infinitesimals, β and γ, namely, $\beta - \gamma$, is of higher order than β or γ. For

$$\frac{\beta - \gamma}{\gamma} = \frac{\beta}{\gamma} - 1;$$

hence

$$\lim \frac{\beta - \gamma}{\gamma} = \lim \left(\frac{\beta}{\gamma} - 1\right)$$

$$= \left(\lim \frac{\beta}{\gamma}\right) - 1 = 0, \qquad \text{q. e. d.}$$

Conversely, if β and γ are two infinitesimals whose difference, $\beta - \gamma$, is of higher order than β or γ, then β and γ are equivalent.

For, since

$$\frac{\beta - \gamma}{\gamma} = \frac{\beta}{\gamma} - 1,$$

it follows that

$$\lim \left(\frac{\beta}{\gamma} - 1\right) = \lim \frac{\beta - \gamma}{\gamma}.$$

The right-hand side of this equation is 0 by hypothesis, and the left-hand side is equal to

$$\left(\lim \frac{\beta}{\gamma}\right) - 1.$$

Hence

$$\lim \frac{\beta}{\gamma} = 1, \qquad \text{q. e. d.}$$

We come now to a theorem of prime importance in the Infinitesimal Calculus.

Fundamental Theorem. *The limit of the ratio of two infinitesimals,*

$$\lim \frac{\beta}{\gamma},$$

is unchanged if the numerator infinitesimal β be replaced by any equivalent infinitesimal β' and the denominator infinitesimal γ be replaced by any equivalent infinitesimal γ'.

In other words:

$$\lim \frac{\beta}{\gamma} = \lim \frac{\beta'}{\gamma'}$$

provided $\qquad \lim \frac{\beta}{\beta'} = 1 \qquad$ *and* $\qquad \lim \frac{\gamma}{\gamma'} = 1.$

The proof is immediate. It is obvious that

$$\frac{\beta'}{\gamma'} = \frac{\beta'\beta\gamma}{\beta\gamma\gamma'}.$$

Hence by Theorem II, Chapter II, § 5 we have

$$\lim \frac{\beta'}{\gamma'} = \left(\lim \frac{\beta'}{\beta}\right)\left(\lim \frac{\beta}{\gamma}\right)\left(\lim \frac{\gamma}{\gamma'}\right).$$

But the first and third limits on the right-hand side are each equal to 1 by hypothesis. Hence

$$\lim \frac{\beta'}{\gamma'} = \lim \frac{\beta}{\gamma}, \qquad \text{q. e. d.}$$

The theorem can be stated in the following equivalent form:

The limit of the ratio of two infinitesimals is the same as the limit of the ratio of their principal parts.

The student must not generalize from this theorem and infer that an infinitesimal can always and for all purposes be replaced by an equivalent infinitesimal. Thus if

$$\beta = 2\alpha + \alpha^3 \qquad \text{and} \qquad \gamma = 2\alpha - \alpha^2,$$

their difference, $\qquad \beta - \gamma = \alpha^3 + \alpha^2,$

is an infinitesimal of the second order. On the other hand,

$$\gamma' = 2\alpha$$

is equivalent to γ. But it is not true that the difference of β and γ', namely,

$$\beta - \gamma' = \alpha^3,$$

is an infinitesimal of the second order. It is obviously of order 3. Thus replacing γ by an equivalent infinitesimal has here changed the order of the difference $\beta - \gamma$.

3. Differentials. Let $\quad y = f(x)$

be a function of x, and let D_xy be its derivative:

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_xy.$$

Let the difference $\Delta y/\Delta x - D_xy$ be denoted by ϵ. Then

$$\frac{\Delta y}{\Delta x} = D_xy + \epsilon,$$

and

(1) $$\Delta y = D_xy\,\Delta x + \epsilon\,\Delta x.$$

Since x is the independent variable, Δx can be taken as the principal infinitesimal. D_xy does not vary with Δx; it is a constant, for we are considering its value at a fixed point $x = x_0$. Since, moreover, D_xy is not in general zero, equation (1) represents Δy as the sum of its principal part, $D_xy\,\Delta x$, and an infinitesimal of higher order, $\epsilon\,\Delta x$.

Definition of a Differential. The expression $D_xy\,\Delta x$ is called the *differential* of the function, and is denoted by dy:

(2) $$dy = D_xy\,\Delta x, \qquad \text{or} \qquad df(x) = D_xf(x)\,\Delta x.$$

(read: "differential y" or "differential $f(x)$" or "dy," etc.).

Thus if $$y = x^2,$$

$$dy = 2x\,\Delta x, \qquad \text{or} \qquad dx^2 = 2x\,\Delta x.$$

Since the definition (2) holds for every function $y=f(x)$, it can be applied to the particular function

$$f(x)=x.$$

Hence

(3) $$dx=D_x x\,\Delta x=\Delta x.$$

But it is not in general true that Δy and dy are equal, since ϵ is in general different from 0. Thus we see that *the differential of the independent variable is equal to the increment of that variable; but the differential of the dependent variable is not in general equal to the increment of that variable.*

By means of (3) equation (2) can now be written in the form

(4) $$dy=D_x y\,dx.$$

Hence

(5) $$\frac{dy}{dx}=D_x y.$$

Geometrically, the increment Δy of the function is represented by the line MP', Fig. 33; and the differential, dy, is equal to MQ, for from (5)

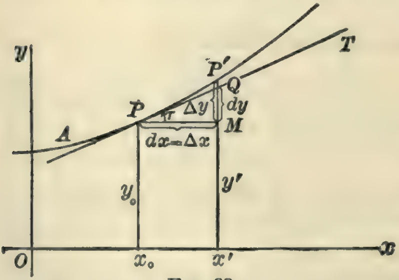


FIG. 33

$$\tan\tau=\frac{dy}{dx}$$

or $$dy=dx\tan\tau.$$

In other words, Δy represents the distance from the level of P to the *curve*, when $x=x'$; dy, the distance from the level of P to the *tangent*. Moreover, the difference

$$\Delta y-dy=\epsilon\Delta x$$

is shown geometrically as the line QP', and is obviously from the figure an infinitesimal of higher order than $\Delta x=PM$.

It is also clear from the figure that Δy and dy are equal when and only when the curve $y=f(x)$ is a straight line; *i.e.*

when $f(x)$ is a linear function,

$$f(x) = ax + b.$$

Hitherto x has been taken as the independent variable, Δx as the principal infinitesimal. We come now to the theorem on which the whole value of differentials for the purpose of performing differentiation depends.

THEOREM. *The relation* (4):

$$dy = D_x y\,dx,$$

is true, even when x and y are both dependent on a third variable, t.

Suppose, namely, that x and y come to us as functions of a third variable, t:

$$(6) \qquad x = \phi(t), \qquad y = \psi(t),$$

and that, when we eliminate t between these two equations, we obtain the function

$$y = f(x).$$

Then dx and dy have the following values, in accordance with the above definition, since t, not x, is now the independent variable, Δt the principal infinitesimal:

$$dy = D_t y\,\Delta t, \qquad dx = D_t x\,\Delta t.$$

We wish to prove that

$$dy = D_x y\,dx.$$

Now by Theorem V of Chap. II, § 5:

$$D_t y = D_x y\,D_t x.$$

Hence, multiplying through by Δt, we get:

$$D_t y\,\Delta t = D_x y \cdot D_t x\,\Delta t,$$

or

$$dy = D_x y\,dx, \qquad \text{q. e. d}$$

With this theorem the explicit use of Theorem V in Chap. II, § 5 disappears, Formula V of that theorem now taking on the form of an algebraic identity:

$$\frac{du}{dx}=\frac{du}{dy}\frac{dy}{dx}.$$

To this fact is due the chief advantage of differentials in the technique of differentiation.

Differentials of Higher Order. It is possible to introduce differentials of higher order by a similar definition:

$$(7) \qquad d^2y = D_x{}^2y\,\Delta x^2, \qquad d^3y = D_x{}^3y\,\Delta x^3, \qquad \text{etc.},$$

x being the independent variable. We should then have by (3)

$$(8) \qquad d^2y = D_x{}^2y\,dx^2 \qquad \text{or} \qquad \frac{d^2y}{dx^2} = D_x{}^2y, \qquad \text{etc.}$$

Unfortunately, however, relation (8) does not continue to hold when x and y both depend on a third variable, t. For example, suppose

$$x = t^2, \qquad y = a + t^2.$$

Then

$$y = a + x.$$

When t is taken as the independent variable, we have according to relation (8):

$$d^2y = D_t{}^2y\,dt^2 = 2\,dt^2;$$

and since

$$dx = 2\,t\,dt,$$

it follows that

$$\frac{d^2y}{dx^2}=\frac{2\,dt^2}{4\,t^2\,dt^2}=\frac{1}{2\,t^2}=\frac{1}{2\,x}.$$

On the other hand, when x is taken as the independent variable, relation (8) becomes

$$d^2y = D_x{}^2y\,dx^2 = 0,$$

and consequently

$$\frac{d^2y}{dx^2}=0.$$

Thus the quotient, $\frac{d^2y}{dx^2}$, is seen to have two entirely distinct values according as t or x is taken as the independent variable. We will agree, therefore, to discard this definition. The notation $\frac{d^2y}{dx^2}$ as meaning D_x^2y is, however, universally used in the Calculus, and so we will accept the definitions

$$\frac{d^2y}{dx^2}=D_x^2y, \qquad \frac{d^3y}{dx^3}=D_x^3y, \qquad \text{etc.},$$

interpreting the left-hand sides of these equations, however, *not as ratios*, but as a *single, homogeneous* (and altogether clumsy!) *notation* for that which is expressed more simply by Cauchy's D.

Remark. The operator D_x shall be written when desired as $\frac{d}{dx}$. Thus

$$D_x\frac{x}{a-x} \qquad \text{appears as} \qquad \frac{d}{dx}\frac{x}{a-x}.$$

Again, the equation

$$D_x^2y=D_x(D_xy)$$

appears as

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}.$$

Finally, the following notation is sometimes used:

$$\frac{d^2y}{dx}=D_x^2y\,dx, \qquad \frac{d^3y}{dx^2}=D_x^3y\,dx, \qquad \text{etc.}$$

4. Technique of Differentiation. Consider, for example, Formula II, Chapter II, § 6:

$$D_x(u+v)=D_xu+D_xv.$$

On writing this formula in terms of differentials, we have

$$\frac{d(u+v)}{dx}=\frac{du}{dx}+\frac{dv}{dx}.$$

Now multiply this equation through by dx:

$$d(u+v) = du + dv.$$

Hence the theorem: *The differential of the sum of two functions is equal to the sum of the differentials of these functions.*

The others of the General Formulas, Chapter II, §§ 6, 7, can be treated in a similar way and lead to corresponding theorems in differentials, embodied in the following important group of formulas.

General Formulas of Differentiation.

$$\text{I.}\qquad d(cu) = c\,du.$$

$$\text{II.}\qquad d(u+v) = du + dv.$$

$$\text{III.}\qquad d(uv) = u\,dv + v\,du.$$

$$\text{IV.}\qquad d\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}.$$

As already explained, Theorem V reduces to an obvious algebraic identity:

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx},$$

and so does not need to be tabulated.

Of the special formulas hitherto considered, only two need be tabulated, namely:

Special Formulas of Differentiation.

$$1.\qquad dc = 0.$$

$$2.\qquad dx^n = nx^{n-1}\,dx.$$

The first of these formulas says that the differential of a constant is zero. The second is valid, not only when x is the independent variable, but when x is any function whatever of the independent variable, t. Thus if

$$(1)\qquad u = \sqrt{1-t}$$

and we set

(2) $$x = 1 - t,$$

equation (1) becomes

(3) $$u = x^{\frac{1}{2}}.$$

Hence $$du = \tfrac{1}{2} x^{-\frac{1}{2}} dx.$$

But $$dx = d1 + d(-t) = 0 - dt,$$

and thus $$du = \frac{-dt}{2\sqrt{1-t}} \qquad \text{or} \qquad \frac{du}{dt} = -\frac{1}{2\sqrt{1-t}}.$$

The student should copy off neatly on a card the size of a postal the General Formulas I–IV, the Special Formulas 1., 2., leaving room for a few further special formulas. All the differentiations of the elementary function of the Calculus are based on these two groups of formulas.

To *differentiate* a function means henceforth to find either its derivative or its differential. Of course, when one of these is known, the other can be found by merely multiplying or dividing by the differential of the independent variable.

We proceed to show by a few typical examples how differentials are used in differentiation.

Example 1. Let $$u = 12 - 5x + 7x^3.$$

To find du.

Take the differential of each side of this equation, and apply at the same time Formula II:

$$du = d(12) + d(-5x) + d(7x^3).$$

By Formula 1, $$d(12) = 0.$$

By Formula I,

$$d(-5x) = -5dx \quad \text{and} \quad d(7x^3) = 7dx^3.$$

Hence $$du = -5dx + 21x^2 dx$$
$$= (-5 + 21x^2)\,dx$$

and $$\frac{du}{dx} = -5 + 21x^2.$$

These steps correspond precisely to the steps the student would take if he were using derivatives, only he would not have written them all out in detail. He would have written down at sight:

$$D_x u = -5 + 21x^2.$$

He can avail himself of the facility he has already acquired and shorten the work as follows. Since

$$du = D_x u\, dx,$$

he can begin by writing

$$du = (\qquad\qquad)dx,$$

and then fill in the parenthesis with the derivative.*

Example 2. Let
$$u = \frac{a^2 - x^2}{a^2 + x^2}.$$

To find du.

By Formula IV we have:

$$du = \frac{(a^2 + x^2)d(a^2 - x^2) - (a^2 - x^2)d(a^2 + x^2)}{(a^2 + x^2)^2}$$

$$= \frac{(a^2 + x^2)(-2\,x\,dx) - (a^2 - x^2)(2\,x\,dx)}{(a^2 + x^2)^2}$$

$$= -\frac{4\,a^2 x\,dx}{(a^2 + x^2)^2};$$

$$\frac{du}{dx} = -\frac{4\,a^2 x}{(a^2 + x^2)^2}.$$

The student would probably prefer to work this example as follows. Remembering that

$$du = D_x u\, dx,$$

* The student must be careful not to omit any differentials. If one term of an equation has a differential as a factor, every term must have a differential as a factor. Such an equation as

$$du = -5 + 21\,x^2$$

is absurd, since the left-hand side is an infinitesimal and the right-hand not. Moreover, there is no such thing as $d_x u$.

begin by writing

$$du = \underline{\qquad\qquad\qquad\qquad} dx,$$

and then fill in the fraction by the old familiar methods of Chapter II.

In the two examples just considered, the processes with differentials correspond precisely to those with derivatives, with which the student is already familiar. This will always be true in any differentiation in which *composite functions* are not involved; *i.e.* whenever, according to our earlier methods, the vanished Theorem V of Chapter II, § 8 was not used. It is in the differentiation of composite functions that the method of differentials presents advantages over the earlier method. We turn in the next paragraphs to such examples.

EXERCISES

Differentiate each of the following functions by the method of differentials, and test the result by the methods of Chapter II.

1. $u = x^3 - 3x + 1$. *Ans.* $du = 3x^2dx - 3dx$.

2. $y = a + bx + cx^2$. *Ans.* $dy = b\,dx + 2cx\,dx$.

3. $w = a^3 - z^3$. *Ans.* $dw = -3z^2dz$.

4. $s = 96t - 16t^2$. *Ans.* $\dfrac{ds}{dt} = 96 - 32t$.

5. $s = v_0t + \frac{1}{2}gt^2$. *Ans.* $\dfrac{ds}{dt} = v_0 + gt$.

6. $u = \dfrac{1-x}{1+x}$. *Ans.* $du = \dfrac{-2\,dx}{(1+x)^2}$.

7. $y = \dfrac{x}{1+x^2}$. *Ans.* $dy = \dfrac{dx - x^2\,dx}{(1+x^2)^2}$.

8. $z = \dfrac{1+x+x^2}{2x}$. *Ans.* $dz = \dfrac{x^2-1}{2x^2}dx$.

9. $u = \dfrac{3 - 2x + x^3}{4 + x^2 - x^3}$.

10. $y = \dfrac{a^4 - x^4}{a^4 + a^2x^2 + x^4}$.

5. Continuation. Differentiation of Composite Functions.

Example 3. Let $u = \sqrt{1 + x + x^2}$.

To find $\dfrac{du}{dx}$.

Here, we begin by computing du. To do this, introduce a new variable, y, setting

$$y = 1 + x + x^2.$$

Then $$u = y^{\frac{1}{2}}.$$

Next, take the differential of each side of this equation. By Special Formula 2 above,

$$du = d\,y^{\frac{1}{2}} = \tfrac{1}{2} y^{-\frac{1}{2}}\, dy.$$

Moreover, $$dy = (1 + 2\,x)dx.$$

Hence $$du = \frac{(1 + 2\,x)dx}{2\sqrt{1 + x + x^2}}$$

and $$\frac{du}{dx} = \frac{1 + 2\,x}{2\sqrt{1 + x + x^2}}.$$

Let the student carry through the above differentiation by the methods of Chapter II and compare his work step by step with the foregoing. He will find that, although the two methods are in substance the same, the method of differentials is simpler in form, *since no explicit use of Theorem* V *here is made.*

*Abbreviated Method.** The solution by differentials can be still further abbreviated by not introducing explicitly a new

* The student should not hasten to take this step himself. He will do well to omit the text that follows till he has worked a score or more of problems in differentiating composite functions as set forth under Example 3, introducing each time explicitly a new variable, as y, z, etc. Not until he comes himself to feel that the abbreviation is an aid, should he attempt to use it.

variable, y. The problem is to find du, when

$$u=(1+x+x^2)^{\frac{1}{2}}.$$

Now, Special Formula 2, as has already been pointed out, holds, not merely when x is the independent variable, but for any *function* whatsoever. It might, for example, equally well be written in the form:

$$d[\phi(x)]^n=n[\phi(x)]^{n-1}d\phi(x).$$

In the present case, then, the *content* of that theorem, — the *essential and complete truth* it contains, — enables us to write down at once the equation:

$$d(1+x+x^2)^{\frac{1}{2}}=\tfrac{1}{2}(1+x+x^2)^{-\frac{1}{2}}d(1+x+x^2).$$

This last differential is computed at sight, and thus the answer is obtained in two steps.

Even these two steps are carried out mentally as a single process, when the student has reached the highest point in the technique of differentiation. He then thinks of the formula:

$$d\sqrt{x}=\frac{dx}{2\sqrt{x}},$$

realizes that it holds, not merely when x is the independent variable, but for any function of x, and so writes down first the easy part of the right-hand side of the equation, thus:

$$d\sqrt{1+x+x^2}=\frac{\qquad}{2\sqrt{1+x+x^2}},$$

carrying in his head the fact that the numerator is the differential of the radicand, *i.e.* $d(1+x+x^2)$. This differentiation he performs mentally, and thus has the final answer with no intermediate work on paper:

$$d\sqrt{1+x+x^2}=\frac{(1+2x)dx}{2\sqrt{1+x+x^2}}.$$

Example 4. The method of differentials is especially useful in the case of implicit functions. Thus, to find the derivative of y with respect to x when

$$x^3 - 3xy + 2y^4 = 1.$$

Take the differential of each side:

$$3x^2\,dx - 3x\,dy - 3y\,dx + 8y^3\,dy = 0.$$

Next, collect the terms in dx by themselves; the others will contain dy as a factor:

$$(3x^2 - 3y)dx + (8y^3 - 3x)dy = 0.$$

Hence $$\frac{dy}{dx} = \frac{3y - 3x^2}{8y^3 - 3x}.$$

EXERCISES

Differentiate the following twelve functions by the method of differentials and also by the methods of Chapter II (in either order), introducing each time *explicitly* the auxiliary variable, if one is used.

1. $u = \sqrt{a^4 + a^2x^2 + x^4}$. *Ans.* $du = \dfrac{(a^2x + 2x^3)dx}{\sqrt{a^4 + a^2x^2 + x^4}}$.

2. $y = \dfrac{1}{\sqrt{1 - x^2}}$. *Ans.* $dy = \dfrac{x\,dx}{(1 - x^2)^{\frac{3}{2}}}$.

3. $u = \dfrac{1}{1 - x}$. *Ans.* $du = \dfrac{dx}{(1 - x)^2}$.

Suggestion. Introduce an auxiliary variable $y = 1 - x$. Then $u = y^{-1}$.

4. $u = \dfrac{1}{(1 - x)^2}$. *Ans.* $\dfrac{du}{dx} = \dfrac{2}{(1 - x)^3}$.

5. $y = \dfrac{1}{1 + x^2}$. *Ans.* $\dfrac{dy}{dx} = \dfrac{-2x}{(1 + x^2)^2}$.

6. $s = \dfrac{a^2}{(a+t)^2}$. *Ans.* $\dfrac{ds}{dt} = -\dfrac{2a^2}{(a+t)^3}$

7. $2x^2 - xy + 4y^2 = 5$. *Ans.* $\dfrac{dy}{dx} = \dfrac{4x-y}{x-8y}$.

8. $xy = a^2$. *Ans.* $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

9. $y^2 = 2mx$. *Ans.* $\dfrac{dy}{dx} = \dfrac{m}{y}$.

10. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Ans.* $\dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$.

11. $2x^2 + 3y^2 = 10$. *Ans.* $\dfrac{dy}{dx} = -\dfrac{2x}{3y}$.

12. $2xy - x + y = 0$. *Ans.* $\dfrac{dy}{dx} = \dfrac{1-2y}{2x+1}$.

The student can work the problems at the end of Chapter II by the method of differentials. For further practice, if desired, the following examples are appended.

13. $u = (x^2+1)\sqrt{x^3-x}$. *Ans.* $\dfrac{du}{dx} = \dfrac{7x^4-2x^2-1}{2\sqrt{x^3-x}}$.

14. $y = (x+2b)(x-b)^2$. *Ans.* $\dfrac{dy}{dx} = 3(x^2-b^2)$.

15. $u = \dfrac{x}{\sqrt{a^2-x^2}}$. *Ans.* $\dfrac{du}{dx} = \dfrac{a^2}{\sqrt{(a^2-x^2)^3}}$.

16. $u = \sqrt{\dfrac{a-x}{x}}$. *Ans.* $\dfrac{du}{dx} = -\dfrac{a}{2x\sqrt{ax-x^2}}$.

17. $u = \dfrac{x-a}{\sqrt{2ax-x^2}}$. *Ans.* $\dfrac{du}{dx} = \dfrac{a^2}{\sqrt{(2ax-x^2)^3}}$.

18. $u = \left(\dfrac{x^2+a^2}{x}\right)^2$ *Ans.* $\dfrac{du}{dx} = 2\dfrac{x^3}{x^4-a^4}$.

19. $z = \left(\dfrac{y^4+b^4}{y^2}\right)^2$. *Ans.* $\dfrac{dz}{dy} = 4\dfrac{y^8-b^8}{y^5}$.

20. $u = \frac{2x^2 + a^2}{x^3}\sqrt{a^2 - x^2}$. *Ans.* $\frac{du}{dx} = -\frac{3a^4}{x^4\sqrt{a^2 - x^2}}$.

21. $u = \sqrt{\frac{x^2 - x + 1}{x^2 + x + 1}}$. *Ans.* $\frac{du}{dx} = \frac{3a^4}{(x^2 + x + 1)\sqrt{x^4 + x^2 + 1}}$.

22. $u = \frac{(x - x^3)^{\frac{4}{3}}}{x^4}$. *Ans.* $\frac{du}{dx} = -\frac{8\sqrt[3]{x - x^3}}{3x^4}$.

23. $u = (x^{\frac{1}{3}} - a^{\frac{1}{3}})^4$. *Ans.* $\frac{du}{dx} = \frac{4(x^{\frac{1}{3}} - a^{\frac{1}{3}})^3}{3x^{\frac{2}{3}}}$.

24. $u = x(x^3 + 5)^{\frac{4}{3}}$. *Ans.* $\frac{du}{dx} = 5(x^3 + 1)(x^3 + 5)^{\frac{1}{3}}$.

25. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$.

CHAPTER V

TRIGONOMETRIC FUNCTIONS

1. Radian Measure. In Trigonometry, the *radian measure* of an angle was introduced, apparently for no good purpose. The reason lies in the *importance for the Calculus* of this new system of measurement, and will become clear in the next paragraph, when we come to differentiate the sine. We will first recall the definition.

Let a circle be described with its centre at the vertex O of the angle; let r denote the length of the radius of the circle and s, that of the intercepted arc. Then the radian measure, θ, of the angle is defined as the ratio s/r:

(1) $$\theta = \frac{s}{r}.$$

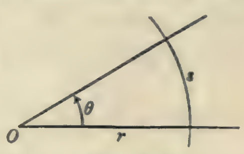


FIG. 34

For a right angle, $s = \dfrac{\pi r}{2}$, and hence $\theta = \dfrac{\pi}{2}$. A straight angle has the measure

$$\theta = \pi = 3.14159\ 26535\ 89793 \cdots.$$

Let ϕ be the measure of the given angle in degrees. Then θ and ϕ are proportional,

$$\theta = c\phi,$$

where c is a constant. To determine c, use a convenient angle whose measure is known in both systems; for example, a straight angle. For the latter,

$$\theta = \pi \qquad \text{and} \qquad \phi = 180.$$

Substituting these values in the above equation we find:

$$\pi = c\,180, \qquad c = \frac{\pi}{180},$$

and hence

$$(2) \qquad \theta = \frac{\pi}{180}\phi, \qquad \phi = \frac{180}{\pi}\theta.$$

This equation can also be written in the form

$$(3) \qquad \frac{\theta}{\pi} = \frac{\phi}{180}$$

and thus an easily remembered rule of conversion from radian measure to degree measure, or the opposite, obtained: *The radian measure of an angle is to π as its degree measure is to* 180.

The unit of angle in radian measure, *i.e.* the angle for which

$$\theta = 1 \qquad \text{and hence} \qquad s = r,$$

is called the *radian.* It is obvious geometrically that it is a little less than 60°. Its precise value (to hundredths of a second) is given by (2):

$$\phi\,|_{\theta=1} = \frac{180}{\pi} = 57°\,17'\,44.81''\ (= 57.29578°).$$

On the other hand, the radian measure of an angle of 1° is

$$\theta\,|_{\phi=1} = \frac{\pi}{180} = .01745 \quad 32925 \quad 19943 \cdots.$$

The student should practice expressing the more important angles, as 30°, 45°, 60°, 90°, 120°, etc., in radian measure until he is thoroughly familiar with the new representation for them.

If, in particular, the radius of the circle is taken as unity, then θ and s are the same number:

$$(4) \qquad \theta = s, \qquad \text{when} \qquad r = 1;$$

or *the arc is equal to the angle.* Thus the radian measure of an angle might have been defined as the length of the intercepted

arc in the unit circle (*i.e.* the circle of unit radius with its centre at O).

Graph of $\sin x$. It is important for the student to make an accurately drawn graph of the function

$$y = \sin x,$$

x being taken in radian measure. Let the unit of length, as usual, be the same on both axes, and let it be chosen as 1 cm. For this purpose Peirce's Table of Integrals (the table of Trigonometric Functions near the end) is especially convenient, since the outside column gives the angles in radian measure, and thus as many points of the graph as are desired can be plotted directly from the tables.

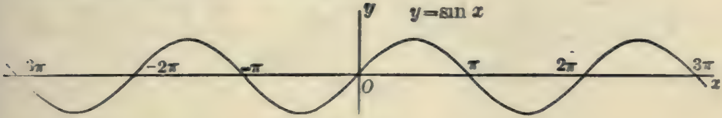


FIG. 35

Since $$\sin(\pi - x) = \sin x$$

each determination of the coordinates (x, y) of a point on the graph, for which $0 < x < \frac{\pi}{2}$ yields at once a second point, namely $(\pi - x, y)$. Thus one arch of the curve is readily constructed from the Tables.*

From this arch a templet, or curved ruler, is made as follows. Lay a card under the arch and with a needle prick through enough points so that the templet can be cut accurately with the scissors.

By means of the templet further arches can be drawn mechanically, and thus the curve is readily continued in both

* The graph could be made directly without tables from purely geometrical considerations. Draw a circle of unit radius. Construct geometrically convenient angles, as those obtained from a right angle by successive bisectors. Measure any one of these angles, $\measuredangle ABP_n$, in radians and this number will be the abscissa of the point on the graph, the

directions to the edges of the paper.* Put this curve in the upper quarter of a sheet of centimetre paper.

The graph brings out clearly the property of the function expressed by the word *periodic.* The function admits the *period* 2π, since

$$\sin(x + 2\pi) = \sin x$$

Graph of $\cos x$. By means of the templet the graph of the function

$$y = \cos x$$

can now be drawn mechanically. This function also admits the period 2π:

$$\cos(x + 2\pi) = \cos x.$$

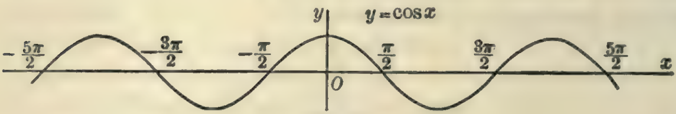


FIG. 37

ordinate being the perpendicular dropped from P_n on the line BA. Thus, if $n = 3$, the coordinates of the point on the graph are:

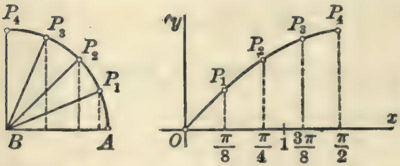


FIG. 36

$$x = \frac{3\pi}{8} = 1.18, \qquad y = .92.$$

A second point of the arch, that corresponding to P_5, has the same y, its coordinate being

$$x = \pi - \frac{3\pi}{8} = 1.96, \qquad y = .92.$$

Of course, the distance π must be laid off on the axis of x by measurement; it cannot be constructed geometrically from the unit length. This done, the further abscissae are found by successive bisectors.

* In order to obtain the most satisfactory figure, observe that the curve has a point of inflection at each of its intersections with the axis of x, the tangent there making an angle of $\pm 45°$ with that axis. Since a curve separates very slowly from an inflectional tangent, it will be well to draw these tangents with a ruler. On laying down the templet, the curve can then be ruled in from the latter with great accuracy. It will not separate sensibly from its tangent for a considerable distance from a point of inflection.

Put the graph in the second quarter of the sheet, choosing the axis of y for this curve in the same vertical line as the axis of y for the sine curve above. There remains the lower half of the sheet for the next graph.

Graph of $\tan x$. The same tables make it easy to plot points profusely on the graph of the function

$$y = \tan x$$

in the interval $0 \leqq x < \frac{\pi}{2}$. Take the axis of y in the same ver-

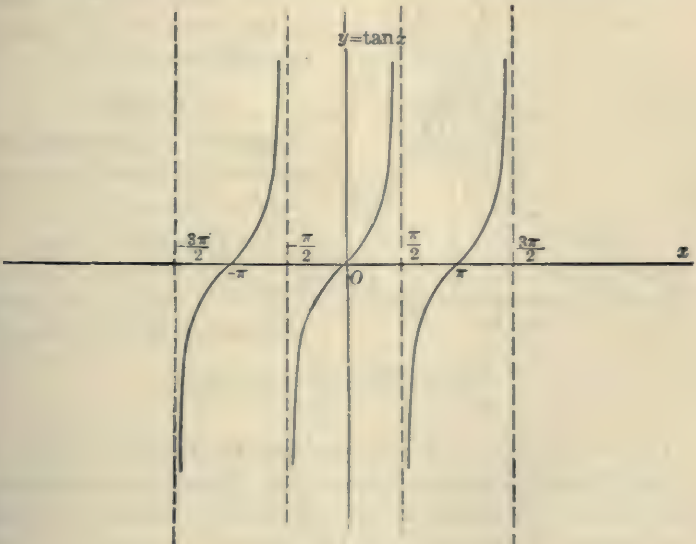


FIG. 38

tical line as in the case of the preceding graphs. This done, a second templet is made and by means of it the graph is drawn mechanically for values of x such that $-\frac{\pi}{2} < x < 0$.

It is desirable furthermore to plot the function in the two adjacent intervals

$$\frac{\pi}{2} < x < \frac{3\pi}{2}, \qquad -\frac{3\pi}{2} < x < -\frac{\pi}{2},$$

in order to suggest the fact that this function admits the period π:

$$\tan(x+\pi) = \tan x.$$

2. Differentiation of $\sin x$. To differentiate the function

(1) $$y = \sin x,$$

apply the definition of a derivative given in Chap. II, § 1.

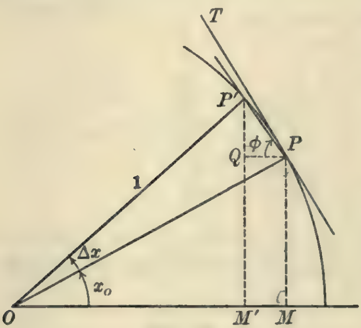


Fig. 39

Give to x an arbitrary value x_0 and compute the corresponding value y_0 of y;

$$y_0 = \sin x_0.$$

Then give x an increment Δx, and compute again the corresponding value of y:

$$y_0 + \Delta y = \sin(x_0 + \Delta x).$$

Hence

$$\Delta y = \sin(x_0+\Delta x) - \sin x_0,$$

(2) $$\frac{\Delta y}{\Delta x} = \frac{\sin(x_0+\Delta x) - \sin x_0}{\Delta x}.$$

It is at this point in the process that the specific properties of the function $\sin x$ come into play. Here, the representation of $\sin x$ by means of the unit circle, familiar from the beginning of Trigonometry, is the key to the solution. From the figure it is clear that

$$\sin x_0 = MP, \qquad \sin(x_0+\Delta x) = M'P',$$

$$\Delta y = \sin(x_0+\Delta x) - \sin x_0 = QP', \qquad \Delta x = \overset{\frown}{PP'}.$$

Hence

(3) $$\frac{\Delta y}{\Delta x} = \frac{QP'}{\overset{\frown}{PP'}},$$

and so we want to know the limit approached by the latter ratio:

$$\lim_{P' \doteq P} \frac{QP'}{\overset{\frown}{PP'}}.$$

By virtue of the Fundamental Theorem of Chap. IV, § 2, we can replace this ratio by a simpler one, since the arc $\overset{\frown}{PP'}$ and the chord $\overline{PP'}$ are equivalent infinitesimals:*

$$\lim \frac{\overline{PP'}}{\overset{\frown}{PP'}} = 1.$$

Hence $$\lim_{P' \doteq P} \frac{QP'}{\overset{\frown}{PP'}} = \lim_{P' \doteq P} \frac{QP'}{\overline{PP'}}.$$

On the other hand, the triangle QPP' is a triangle of reference for the $\measuredangle\, QPP' = \phi$, and so

$$\frac{QP'}{\overline{PP'}} = \sin \phi.$$

When P' approaches P, the secant PP' (*i.e.* the indefinite line determined by the two points P and P') approaches the tangent PT at P, and thus

$$\lim_{P' \doteq P} \phi = \measuredangle\, QPT = \frac{\pi}{2} - x_0.$$

Finally, then,

$$\lim_{P' \doteq P} \frac{QP'}{\overline{PP'}} = \lim_{P' \doteq P} \sin \phi = \sin\left(\frac{\pi}{2} - x_0\right) = \cos x_0,$$

and consequently $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \cos x_0,$$

* The student should assure himself of the truth of this statement by visualizing the figure (making an accurate drawing with ruler and compass for angles of 30°, 15°, and 7½°, the circle used being 10 in. in diameter) and realizing that, when P' is near P, the difference in length between the arc and the chord is but a minute per cent of the length of either one. A formal proof will be found below.

or, on dropping the subscript,

$$D_x \sin x = \cos x. \tag{4}$$

This theorem gives rise to the following theorem in differentials:

$$d \sin x = \cos x\, dx. \tag{5}$$

Reason for the Radian. The reason for measuring angles in terms of the radian as the unit now becomes clear. Had we used the degree, the increment Δx would not have been equal to $\overset{\frown}{PP'}$; we should have had:

$$\frac{\Delta x}{360} = \frac{\overset{\frown}{PP'}}{2\pi}, \qquad \text{or} \qquad \Delta x = \frac{180}{\pi}\overset{\frown}{PP'}.$$

Hence (3) would have read:

$$\frac{\Delta y}{\Delta x} = \frac{\pi}{180} \cdot \frac{QP'}{\overset{\frown}{PP'}},$$

and thus the formula of differentiation would have become:

$$D_x \sin x = \frac{\pi}{180} \cos x.$$

The saving of labor in not being obliged to multiply by this constant each time we differentiate is great. Still more important, however, is the elimination of a multiplier which is of the nature of an extraneous constant, whose presence would have obscured the essential simplicity of the formulas of the Calculus.

EXERCISE

Prove in a similar manner that

$$D_x \cos x = -\sin x.$$

3. Certain Limits. In the foregoing paragraph we have made use of the fact that *the ratio of the arc to the chord approaches 1 as its limit.* A formal proof of this theorem, based on the

axioms of geometry, can be given as follows. Draw the tangent at P and erect a perpendicular at P' cutting the tangent in Q. Denote the angle $\measuredangle\, P'PQ$ by α.

Then
$$\overline{PP'} < \overset{\smile}{PP'} < PQ + P'Q;$$

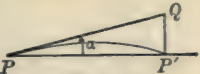


FIG. 40

for i) a straight line is the shortest distance between two points; and ii) a convex curved line is less than a convex broken line which envelops it and has the same extremities. But

$$PQ = \frac{\overline{PP'}}{\cos\alpha}, \qquad P'Q = \overline{PP'}\tan\alpha.$$

Hence
$$1 < \frac{\overset{\smile}{PP'}}{\overline{PP'}} < \frac{1}{\cos\alpha} + \tan\alpha.$$

When α approaches 0, the right-hand member of the double inequality approaches 1; hence the middle member must also approach 1, or

$$\lim \frac{\overset{\smile}{PP'}}{\overline{PP'}} = 1, \qquad \text{q. e. d.}$$

The foregoing proof holds, not merely for a circle, but for any curve with a convex arc $\overset{\smile}{PP'}$. Consequently the theorem is established generally.

The Limit $\lim\limits_{\alpha \doteq 0} \dfrac{\sin\alpha}{\alpha}$. From Fig. 41 it is clear that

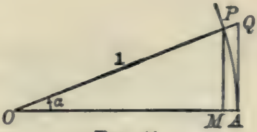


FIG. 41

$$MP = \sin\alpha, \qquad \overset{\smile}{AP} = \alpha,$$

and hence
$$\frac{\sin\alpha}{\alpha} = \frac{MP}{\overset{\smile}{AP}}.$$

By direct inspection of the figure it is seen, then, that

(1)
$$\lim_{\alpha \doteq 0} \frac{\sin\alpha}{\alpha} = 1.$$

A formal proof of this equation can be given as follows From Fig. 42

$$\overline{PP'} = 2\sin\alpha, \qquad \overset{\frown}{PP'} = 2\alpha.$$

Hence

$$\frac{\sin\alpha}{\alpha} = \frac{\overline{PP'}}{\overset{\frown}{PP'}},$$

and therefore, by the proposition just established,

$$\lim_{\alpha \doteq 0} \frac{\sin\alpha}{\alpha} = \lim_{P' \doteq P} \frac{\overline{PP'}}{\overset{\frown}{PP'}} = 1.$$

Another Proof of (1). The area of the sector OAP, Fig. 42, is $\frac{1}{2}\alpha$, and it obviously lies between the areas of the triangles OMP and OPN. Hence

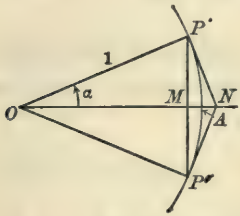


FIG. 42

$$\tfrac{1}{2}\sin\alpha\cos\alpha < \tfrac{1}{2}\alpha < \tfrac{1}{2}\tan\alpha$$

or

$$\cos\alpha < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha}.$$

When α approaches 0, each of the extreme terms approaches 1, and so the middle term must also do so, q. e. d.

From Peirce's *Tables*, p. 130, we see that

$$\sin 4^\circ 40' = .0814,$$

and the same angle, measured in radians, also has the value .0814, to three significant figures. Thus for values of α not exceeding .0814, $\sin\alpha$ differs from α by less than one part in 800, or one-eighth of one per cent.

The Limits $\displaystyle\lim_{\alpha \doteq 0} \frac{1-\cos\alpha}{\alpha}$ *and* $\displaystyle\lim_{\alpha \doteq 0} \frac{1-\cos\alpha}{\alpha^2}$. From Fig. 42, the first of these limits is seen to have the value 0:

$$\lim_{\alpha \doteq 0} \frac{1-\cos\alpha}{\alpha} = 0.$$

A formal proof can be derived at once by the method employed in the evaluation of the next limit,

$$\lim_{\alpha \doteq 0} \frac{1-\cos\alpha}{\alpha^2}.$$

Expressing $1-\cos\alpha$ in terms of the half angle, we have

$$1-\cos\alpha = 2\sin^2\frac{\alpha}{2}.$$

Hence
$$\frac{1-\cos\alpha}{\alpha^2} = \frac{2\sin^2\frac{\alpha}{2}}{\alpha^2} = \frac{1}{2}\left[\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\right]^2$$

and
$$\lim_{\alpha \doteq 0} \frac{1-\cos\alpha}{\alpha^2} = \frac{1}{2}\lim_{\alpha \doteq 0}\left[\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\right]^2 = \frac{1}{2}.$$

EXERCISES

In the accompanying figure determine the following limits when α approaches 0:

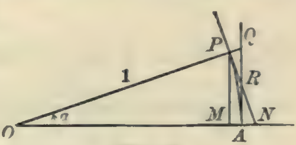


FIG. 43

1. $\lim \frac{AR}{MP}$. *Ans.* $\frac{1}{2}$.

2. $\lim \frac{AQ}{\overset{\frown}{AP}}$. *Ans.* 1.

3. $\lim \frac{RQ}{MP}$.

4. $\lim \frac{RP}{\overset{\frown}{AP}}$.

5. $\lim \frac{PN}{\overline{AP}}$.

6. $\lim \frac{MA}{PQ}$.

7. $\lim \frac{PQ}{AN}$.

8. $\lim \frac{RQ}{PN}$.

Determine the principal part of each of the following infinitesimals, referred to α as principal infinitesimal:

9. MP. *Ans.* α.

10. PR. *Ans.* $\frac{1}{2}\alpha$.

11. RQ.

12. PN.

13. AQ.

14. MA. *Ans.* $\frac{1}{2}\alpha^2$

15. PQ.

16. MN.

17. $AQ - MP$.

4. Critique of the Foregoing Differentiation. The differentiation of $\sin x$ as given in § 1 has the advantage of being direct and lucid, and thus easily remembered. Each analytic step is mirrored in a simple geometric construction. It has the disadvantage, however, of incompleteness. For, first, we have allowed Δx, in approaching 0, to pass only through positive values; and secondly we have assumed x_0 to lie between 0 and $\frac{1}{2}\pi$. Hence there are in all seven more cases to consider.

An analytic method that is simple and at the same time general is the following. Recall the Addition Theorem for the sine:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b,$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b,$$

whence $$\sin(a+b) - \sin(a-b) = 2\cos a \sin b.$$

Let $$a+b = x_0 + \Delta x, \qquad a - b = x_0.$$

Solving these last equations for a and b, we get:

$$a = x_0 + \frac{\Delta x}{2}, \qquad b = \frac{\Delta x}{2}.$$

Thus $$\sin(x_0 + \Delta x) - \sin x_0 = 2\cos\left(x_0 + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2},$$

and the difference-quotient becomes

$$\frac{\Delta y}{\Delta x} = \cos\left(x_0 + \frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}.$$

The first factor on the right approaches the limit $\cos x_0$ when Δx approaches 0. On setting $\frac{1}{2}\Delta x = \alpha$, the second factor becomes

$$\frac{\sin\alpha}{\alpha}.$$

Hence the factor approaches 1. Thus

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \cos x_0,$$

or, on dropping the subscript,

$$D_x \sin x = \cos x.$$

5. Differentiation of $\cos x$, $\tan x$, **etc.** To differentiate the function $\cos x$, introduce a new variable, y, by the equation

$$y = \frac{\pi}{2} - x. \qquad \text{Hence} \qquad x = \frac{\pi}{2} - y,$$

and
$$\cos x = \cos\left(\frac{\pi}{2} - y\right) = \sin y.$$

Taking the differential of each side of the equation thus obtained, we have:

$$d\cos x = d\sin y = \cos y\,dy.$$

But
$$\cos y = \sin x \qquad \text{and} \qquad dy = -\,dx.$$

Hence

(1)
$$d\cos x = -\sin x\,dx.$$

To differentiate the function $\tan x$, set

$$\tan x = \frac{\sin x}{\cos x}.$$

Hence
$$d\tan x = \frac{\cos x\,d\sin x - \sin x\,d\cos x}{\cos^2 x}$$

$$= \frac{\cos^2 x\,dx + \sin^2 x\,dx}{\cos^2 x} = \frac{dx}{\cos^2 x},$$

and thus

(2)
$$d\tan x = \sec^2 x\,dx.$$

It is shown in a similar manner (or by setting $x = \frac{\pi}{2} - y$ in the equation just deduced) that

(3)
$$d\cot x = -\csc^2 x\,dx.$$

These are the important formulas of differentiation for the trigonometric functions. By means of them all other differentiations of these functions can be readily performed. Thus,

to differentiate the function $\sec x$, set

$$\sec x = (\cos x)^{-1}.$$

Then $$d \sec x = -\frac{d \cos x}{\cos^2 x} = \frac{\sin x\, dx}{\cos^2 x}.$$

It is not desirable to tabulate the result, since one rarely has occasion to differentiate either $\sec x$ or $\csc x$, and when the occasion does arise, the differentiation can be worked out directly, as above.

The student should now add to his card of Special Formulas the four main formulas just obtained. This card will now read as follows:

1. $$dc = 0.$$
2. $$d\,x^n = nx^{n-1}\,dx.$$
3. $$d \sin x = \cos x\, dx.$$
4. $$d \cos x = -\sin x\, dx.$$
5. $$d \tan x = \sec^2 x\, dx.$$
6. $$d \cot x = -\csc^2 x\, dx.$$

6. Shop Work. To acquire facility in the use of the new results, the student should work a generous number of simple examples, for which the following are typical.

Example 1. To differentiate the function

$$u = \sin ax.$$

Let $$y = ax.$$

Then $$u = \sin y,$$

and $$du = d \sin y = \cos y\, dy.$$

But $$dy = a\, dx.$$

Hence, substituting, we have

$$du = a \cos ax\, dx \qquad \text{or} \qquad \frac{d}{dx} \sin ax = a \cos ax.$$

The solution can be abbreviated as follows. The equation

$$d \sin x = \cos x \, dx$$

is true, not merely when x is the independent variable. It holds, for example, in the form

$$d \sin y = \cos y \, dy,$$

where y is any function of x. Hence we can write immediately

$$d \sin ax = \cos ax \, d(ax),$$

and thus obtain the result

$$d \sin ax = a \cos ax \, dx.$$

Example 2. To differentiate the function

$$u = \sqrt{1 - k^2 \sin^2 \phi}.$$

Let $$z = 1 - k^2 \sin^2 \phi.$$

Then $$u = z^{\frac{1}{2}}$$

$$du = d\,z^{\frac{1}{2}} = \tfrac{1}{2} z^{-\frac{1}{2}} \, dz\,;$$

$$dz = -k^2 d \sin^2 \phi.$$

Let $$y = \sin \phi.$$

Then $$dy = \cos \phi \, d\phi$$

and $$d \sin^2 \phi = d\,y^2 = 2y\,dy = 2 \sin \phi \cos \phi \, d\phi.$$

Hence $$du = \tfrac{1}{2} z^{-\frac{1}{2}}(-2k^2 \sin \phi \cos \phi \, d\phi)$$

or $$\frac{du}{d\phi} = -\frac{k^2 \sin \phi \cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

Example 3. If $$\sin x + \sin y = x - y,$$

to find $\frac{dy}{dx}$. Take the differential of each side of the equation·

$$\cos x \, dx + \cos y \, dy = dx - dy.$$

Hence $$(\cos x - 1)dx + (\cos y + 1)dy = 0$$

and $$\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos y}.$$

EXERCISES

Differentiate the following functions.

1. $u = \cos ax.$ $\qquad \dfrac{du}{dx} = -a \sin ax.$

2. $y = \cos^2 x.$ $\qquad \dfrac{dy}{dx} = -2 \sin x \cos x$

3. $y = \csc x.$ $\qquad \dfrac{dy}{dx} = -\csc^2 x \cos x.$

4. $u = \tan \dfrac{x}{2}.$ $\qquad \dfrac{du}{dx} = \dfrac{1}{2} \sec^2 \dfrac{x}{2}.$

5. $u = \cot 2x.$ $\qquad \dfrac{du}{dx} = -2 \csc^2 2x.$

6. $u = \sec 3x.$

7. $u = \tan^2 ax.$

8. $u = \sin^3 x.$

9. $u = 1 - \sin x.$

10. $u = x + \tan x.$

11. $u = \cos^3 x.$

12. $u = \sec^2 x.$

13. $u = \sin x \cos x.$

14. $u = \dfrac{\sin x}{1 - \cos x}.$ $\qquad \dfrac{du}{dx} = -\dfrac{1}{2} \csc^2 \dfrac{x}{2}.$

15. $u = \sqrt{1 + \cos x}.$ $\qquad \dfrac{du}{dx} = -\dfrac{1}{\sqrt{2}} \sin \dfrac{x}{2}.$

16. $u = \dfrac{1 - \cos x}{1 + \cos x}.$

17. $u = \dfrac{1 + \sin x}{1 - \sin x}.$

18. $u = \dfrac{\sin x}{a + b \cos x}.$

19. $u = \dfrac{1}{a \cos x + b \sin x}.$

20. $u = \dfrac{1}{\sin x + \cos x}.$

21. $u = \dfrac{1}{(a + b \cos x)^2}.$

22.* $u = \text{vers } x.$ $\qquad \dfrac{du}{dx} = \sin x.$

23.* $u = \text{covers } x.$ $\qquad \dfrac{du}{dx} = -\cos x.$

* The *versed sine* and the *coversed sine* are defined as follows:

$$\text{vers } x = 1 - \cos x; \qquad \text{covers } x = 1 - \sin x.$$

24. $u = x \sin 2x.$ **25.** $u = \dfrac{\cos \dfrac{x}{2}}{x}.$

26. $u = \tan\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right).$ **27.** $u = \cot\left(\dfrac{x}{2} - \dfrac{\pi}{4}\right).$

28. $u = \tan \dfrac{x}{1-x}.$ **29.** $u = \dfrac{\sin \pi x}{x}.$

30. $u = \sin x + \cos 2x.$ **31.** $u = x^2 \cos \pi x.$

32. $u = \dfrac{1}{\sqrt{1 - k^2 \sin^2 \phi}}.$ **33.** $u = \dfrac{\cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$

34. $x \cos y = \sin(x + y).$ $\dfrac{dy}{dx} = -\dfrac{\cos(x+y) - \cos y}{\cos(x+y) + x \sin y}.$

35. $\tan x - \cot y = \sin x \sin y.$ **36.** $\sin x + \sin y = 1.$

37. $\tan \theta + \tan \phi = 2 \tan \phi \tan \theta.$ **38.** $x = y \sin y.$

7. Maxima and Minima. By means of the new functions studied in this chapter the range of problems in maxima and minima which can be treated by the Calculus has been materially enlarged. No new principles are involved; the student should go over carefully the paragraphs of Chap. III relating to this subject, before he proceeds farther with the present paragraph.

Example 1. A man in a rowboat 1 mile off shore wishes to go to a point which is 2 miles inland and 4 miles up the beach. If he can row at the rate of 5 miles an hour, but can walk only 3 miles an hour after he lands, in what direction should he row in order to get to his destination in the shortest possible time?

In the first place, it is clear that the straight line AEB is *not* the best path. For, if he rows toward a point P slightly farther up the beach, the amount by which he lengthens the leg AP of his path is very nearly equal to the amount by which

he shortens the leg PB.* Consequently the time is shortened.

On the other hand, P obviously ought not to be taken so far up the beach as D. The minimum occurs, therefore, for some intermediate point.

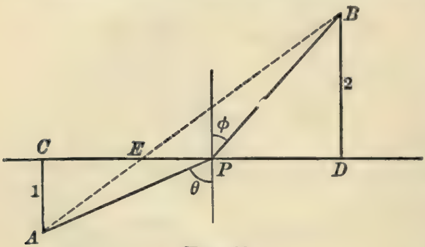


FIG. 44

Let the angles θ, ϕ be taken as indicated in the figure. Then, since $t = \frac{s}{v}$,

$$\text{time from } A \text{ to } P = \frac{AP}{5} = \frac{1}{5\cos\theta};$$

$$\text{time from } P \text{ to } B = \frac{PB}{3} = \frac{2}{3\cos\phi}.$$

Hence the total time, u, which is to be made a minimum is

$$(1) \qquad u = \frac{1}{5\cos\theta} + \frac{2}{3\cos\phi}.$$

Moreover, θ and ϕ are connected with each other by a relation which is readily obtained by expressing the distance CD in two ways:

$$(2) \qquad \tan\theta + 2\tan\phi = 4.$$

We are now ready to compute $du/d\theta$ and set it equal to 0:

$$du = -\frac{d\cos\theta}{5\cos^2\theta} - \frac{2\,d\cos\phi}{3\cos^2\phi}$$

$$= \frac{\sec^2\theta\sin\theta}{5}\,d\theta + \frac{2\sec^2\phi\sin\phi}{3}\,d\phi;$$

$$(3) \qquad \frac{du}{d\theta} = \frac{\sec^2\theta\sin\theta}{5} + \frac{2\sec^2\phi\sin\phi}{3}\frac{d\phi}{d\theta}.$$

* Let the student not leave this statement till he is absolutely convinced of its truth. An accurate figure on a large scale will bring the fact out clearly.

On setting $du/d\theta = 0$, we obtain the equation:

$$\text{(4)} \qquad \frac{\sec^2\theta \sin\theta}{5} = -\frac{2\sec^2\phi \sin\phi}{3}\frac{d\phi}{d\theta}.$$

Next, differentiate (2):

$$\sec^2\theta d\theta + 2\sec^2\phi d\phi = 0,$$

or

$$\text{(5)} \qquad \sec^2\theta = -2\sec^2\phi \frac{d\phi}{d\theta}.$$

Now, divide equation (4) by equation (5): *

$$\text{(6)} \qquad \frac{\sin\theta}{5} = \frac{\sin\phi}{3} \qquad \text{or} \qquad \frac{\sin\theta}{\sin\phi} = \frac{5}{3}.$$

The result, stated in words, is as follows: $\sin\theta$ *is to* $\sin\phi$ *as the velocity in water is to the velocity on land.*

Let the student work the general problem, in which all the data are taken in literal form, and verify the general result just stated.

In order actually to determine θ, equations (2) and (6) must be solved as simultaneous:

$$\text{(7)} \qquad \begin{cases} \tan\theta + 2\tan\phi = 4, \\ 3\sin\theta = 5\sin\phi. \end{cases}$$

This is done best by the method of Trial and Error, as it is called in Physics; Successive Approximations being the name usually given to it in Mathematics. It is a most important method in both sciences, and the student should let no opportunity go by to use the method whenever, as here, he meets a case which calls for it. Cf. Chap. VII, § 5.

The Corresponding Problem in Optics. We have stated and solved a problem which is not lacking in interest, but which appears to have no scientific importance. This very problem, however, occurs in Optics. The velocity of light is different in

* *i.e.* divide the left-hand side of (4) by the left-hand side of (5) for a new left-hand side; and do the same thing for the right-hand sides.

different media, such as air and water. Suppose two media to be in contact with each other, the common boundary being a plane. Let A be a luminous point, from which rays emanate in all directions. When the rays strike the bounding surface, they are all refracted and enter the second medium in case the velocity of light in that medium is less than in the first. One of the refracted rays will pass through a given point B. And now the law of light is that the time required for the light to pass from A to B is less for this path than for any other possible path.

If, then, the velocity of light in the first medium is u * and in the second medium, v, we have:

$$\frac{\sin\theta}{\sin\phi}=\frac{u}{v}=n,$$

where n is the *index of refraction* for the passage from the first medium into the second.

EXERCISES

1. A wall 27 ft. high is 64 ft. from a house. Find the length of the shortest ladder that will reach the house if one end rests on the ground outside the wall.

Take the angle which the ladder makes with the horizontal as the independent variable.

2. The equal sides of an isosceles triangles are each 8 in. long, the base being variable. Show that the triangle of maximum area is the one which has a right angle.

Take one of the base angles as the independent variable, ϕ.

3. A gutter is to be made out of a long strip of copper 9 in. wide by bending the strip along two lines parallel to the edges and distant respectively 3 in. from an edge. Thus the cross-section will be a broken line, made up of three straight lines, each 3 in. long. How wide should the gutter be at the

* The letter u used here has nothing to do with the u used above in solving the problem.

top, in order that its carrying capacity may be as great as possible? *Ans.* 6 in.

4. Johnny is to have a piece of pie, the perimeter of which is to be 12 in. If Johnny may choose the plate on which the pie is to be baked, what size plate would he naturally select?

5. A can-buoy in the form of a double cone is to be made from two equal circular iron plates by cutting out a sector from each plate and bending up the plate. If the radius of each plate is a, find the radius of the base of the cone when the buoy is as large as possible. *Ans.* $a\sqrt{\frac{2}{3}}$.

6. From a circular piece of filter paper a sector is to be cut and then bent into the form of a cone of revolution. Show that the largest cone will be obtained if the angle of the sector is .8165 of four right angles.

7. Two solid spheres, whose diameters are 8 in. and 18 in., have their centres 35 in. apart. At what point in their line of centres and between the spheres should a light be placed in order to illuminate the largest amount of spherical surface?

Ans. 8 in. from the centre of the smaller sphere.

8. Find the most economical proportions for a conical tent.

9. A block of stone is to be drawn along the floor by a rope. Find the angle which the rope should make with the horizontal in order that the tension may be as small as possible.

Ans. The angle of friction.

10. A block of stone is to be drawn up an inclined plane by a rope. Find the angle which the rope should make with the plane, in order that the tension in the rope be as small as possible.

11. A statue ten feet high stands on a pedestal that is 50 ft. high. How far ought a man whose eyes are 5 ft. above the ground to stand from the pedestal in order that the statue may subtend the greatest possible angle?

12. A steel girder 25 ft. long is moved on rollers along a passageway 12.8 ft. wide, and into a corridor at right angles

to the passageway. Neglecting the horizontal width of the girder, find how wide the corridor must be in order that the girder may go round the corner. *Ans.* 5.4 ft.

13. A gutter whose cross-section is an arc of a circle is to be made by bending into shape a strip of copper. If the width of the strip is a, find the radius of the cross-section when the carrying capacity of the gutter is a maximum. *Ans.* a/π.

14. A long strip of paper 8 in. wide is cut off square at one end. A corner of this end is folded over on to the opposite side, thus forming a triangle. Find the area of the smallest triangle that can thus be formed.

15. In the preceding question, when will the length of the crease be a minimum?

16. The captain of a man-of-war saw, one dark night, a privateersman crossing his path at right angles and at a distance ahead of c miles. The privateersman was making a miles an hour, while the man-of-war could make only b miles in the same time. The captain's only hope was to cross the track of the privateersman at as short a distance as possible under his stern, and to disable him by one or two well-directed shots; so the ship's lights were put out and her course altered in accordance with this plan. Show that the man-of-war crossed the privateersman's track $\frac{c}{b}\sqrt{a^2 - b^2}$ miles astern of the latter.

If $a = b$, this result is absurd. Explain.

17. The illumination of a small plane surface by a luminous point is proportional to the cosine of the angle between the rays of light and the normal to the surface, and inversely proportional to the square of the distance of the luminous point from the surface. At what height on the wall should an arc light be placed in order to light most brightly a portion of the floor a ft. distant from the wall?

Ans. About $\frac{7}{10}a$ ft. above the floor

18. A town A situated on a straight river, and another town B, a miles farther down the river and b miles back from the river, are to be supplied with water from the river pumped by a single station. The main from the waterworks to A will cost \$ m per mile and the main to B will cost \$ n per mile. Where on the river-bank ought the pumps to be placed?

19. A telegraph pole 25 ft. high is to be braced by a stay 20 ft. long, one end of the stay being fastened to the pole and the other end to a short stake driven into the ground. How far from the pole should the stake be located, in order that the stay be most effective?

20. Into a full conical wine-glass whose depth is a and generating angle α there is carefully dropped a spherical ball of such a size as to cause the greatest overflow. Show that the radius of the ball is

$$\frac{a \sin \alpha}{\sin \alpha + \cos 2\alpha}.$$

21. A foot-ball field $2a$ ft. long and $2b$ ft. broad is to be surrounded by a running track consisting of two straight sides (parallel to the length of the field) joined by semicircular ends. The track is to be $4c$ ft. long. Show how it should be made in order that the shortest distance between the track and the foot-ball field may be as great as possible.

22.* The number of ems (or the number of sq. cms. of text) on this page and the breadths of the margins being given, what ought the length and breadth of the page to be that the amount of paper used may be as small as possible?

23. Assuming that the values of diamonds are proportional, other things being equal, to the squares of their weights, and that a certain diamond which weighs one carat is worth \$ m, show that it is safe to pay at least \$ $8m$ for two diamonds which together weigh 4 carats, if they are of the same quality as the one mentioned.

* Exs. 22–25 do not involve Trigonometry.

24. When a voltaic battery of given electromotive force (E volts) and given internal resistance (r ohms) is used to send a steady current through an external circuit of R ohms resistance, an amount of work, W, equivalent to

$$\frac{E^2R}{(r+R)^2} \times 10^7 \text{ ergs}$$

is done each second in the outside circuit. Show that, if different values be given to R, W will be a maximum when $R = r$.

25. An ice cream cone is to hold one-eighth of a pint. The slant height is l, and half the angle at the vertex is x. Find the value of x that will make the cost of manufacture of the cone a minimum. (*Ans.* $x = 35°.27$.)

8. Tangents in Polar Coordinates. Let

$$r = f(\theta)$$

be the equation of a curve in polar coordinates. We wish to find the direction of its tangent. The direction will be known if we can determine the angle ψ between the radius vector produced and the tangent. Let P, with the coordinates (r_0, θ_0), be an arbitrary point of the curve and $P': (r_0 + \Delta r,\ \theta_0 + \Delta\theta)$ a neighboring point. Draw the chord PP' and denote the $\angle\, OP'P$ by ψ'. Then obviously

$$\lim_{P' \doteq P} \psi' = \psi_0.$$

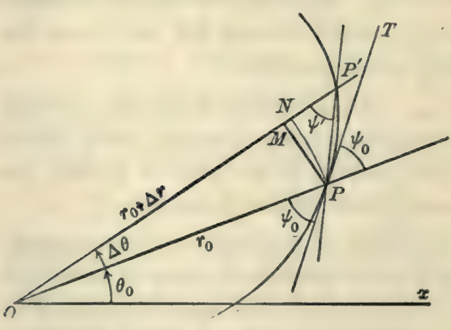


FIG. 45

To determine ψ_0, drop a perpendicular PM from P on the radius vector OP' and draw an arc PN of a circle with O as centre. The right triangle $MP'P$ is a triangle of refer-

ence for the angle ψ' and

$$\cot \psi' = \frac{P'M}{MP}.$$

Hence

(1) $$\cot \psi_0 = \lim_{P' \doteq P} \cot \psi' = \lim_{P' \doteq P} \frac{P'M}{MP}.$$

In the latter ratio we can replace $P'M$ and MP by more convenient infinitesimals; cf. Chap. IV, § 2. We observe that

$$MP = r_0 \sin \Delta\theta; \qquad \text{hence} \qquad \lim_{\Delta\theta \doteq 0} \frac{MP}{r_0 \Delta\theta} = \lim_{\Delta\theta \doteq 0} \frac{\sin \Delta\theta}{\Delta\theta} = 1\cdot$$

i.e. MP and $r_0\Delta\theta$ are equivalent infinitesimals.

Furthermore, $P'M$ and $P'N = \Delta r$ are also equivalent infinitesimals. For

$$P'M = P'N + NM$$

and

$$NM = r_0 - r_0 \cos \Delta\theta.$$

Hence

$$\frac{NM}{\Delta r} = \frac{r_0 \dfrac{1 - \cos \Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta}}.$$

Now, by § 3,

$$\lim_{\Delta\theta \doteq 0} \frac{1 - \cos \Delta\theta}{\Delta\theta} = 0.$$

On the other hand,

$$\lim_{\Delta\theta \doteq 0} \frac{\Delta r}{\Delta\theta} = D_\theta r,$$

and this quantity is not, in general, 0. Hence

$$\lim_{\Delta\theta \doteq 0} \frac{NM}{\Delta r} = 0.$$

Returning to equation (1) we can now write the last limit in the form:

$$\lim_{P' \doteq P} \frac{P'M}{MP} = \lim_{\Delta\theta \doteq 0} \frac{\Delta r}{r_0 \Delta\theta} = \frac{1}{r_0} D_\theta r;$$

or, dropping subscripts,

(2) $$\cot \psi = \frac{1}{r} D_\theta r.$$

In terms of differentials, this result can be written in either of the two forms:

(3) $$\cot\psi = \frac{dr}{r\,d\theta}, \qquad \tan\psi = \frac{r\,d\theta}{dr}.$$

Example. Consider the parabola in polar form:

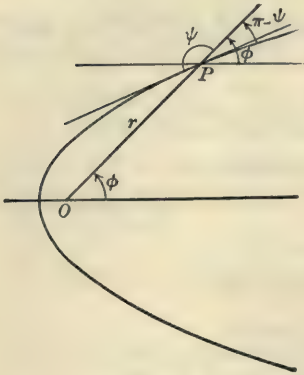


FIG. 46

$$r = \frac{m}{1-\cos\phi}.$$

To determine ψ. Here,

$$dr = -\frac{m\sin\phi\,d\phi}{(1-\cos\phi)^2}.$$

Hence

$$\cot\psi = -\frac{m\sin\phi\,d\phi}{(1-\cos\phi)^2}\cdot\frac{1-\cos\phi}{m\,d\phi}$$

$$= -\frac{\sin\phi}{1-\cos\phi}.$$

In particular, at the extremity of the latus rectum, we have:

$$\cot\psi\,\Big|_{\phi=\frac{\pi}{2}} = -1, \qquad \psi = \frac{\pi}{4}+\frac{\pi}{2},$$

and thus we obtain anew the result that the tangent there makes an angle of 45° with the axis of the parabola.

Again, at the vertex,

$$\cot\psi\,|_{\phi=\pi} = 0, \qquad \psi = \frac{\pi}{2},$$

and the tangent there is verified as perpendicular to the axis

From the above equation,

$$\cot\psi = -\frac{\sin\phi}{1-\cos\phi},$$

a simple relation between ψ and ϕ can be deduced. Since

$$\frac{\sin\phi}{1-\cos\phi} = \frac{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\sin^2\frac{\phi}{2}} = \cot\frac{\phi}{2},$$

it follows that
$$\cot\psi = -\cot\frac{\phi}{2}.$$

But, for any angle, x,
$$\cot(\pi - x) = -\cot x.$$

Setting $x = \psi$ in the above equation, we have:
$$\cot(\pi - \psi) = \cot\frac{\phi}{2}.$$

Hence *
$$\pi - \psi = \frac{\phi}{2},$$

or, *the supplement of* ψ *is equal to* $\frac{\phi}{2}$. Thus we have a new proof of the familiar property of the parabola, that the tangent at any point P of the curve bisects the angle between the focal radius, OP, and a parallel to the axis, drawn through P.

EXERCISES

1. Plot the spiral, $r = \theta$,

and show that the angle at which it crosses the prime direction when $\theta = 2\pi$ is $80° \, 57'$.

2. Plot the spiral, $r = \frac{1}{\theta}$.

Show that it has an asymptote parallel to the prime vector.

Suggestion. Consider the distance of a point P of the curve from the prime direction, and find the limit of this distance when θ approaches 0.

Determine the angle at which the radius vector corresponding to $\theta = \pi/2$ meets this curve.

3. Plot the cardioid,
$$r = a(1 - \cos\phi),$$

* The trigonometric equation admits a second solution, namely $(\pi - \psi) + \pi = \phi/2$. If, however, we agree to take ϕ and ψ so that $0 \leqq \phi < 2\pi$ and $0 \leqq \psi < \pi$, this second solution is ruled out.

and show that
$$\cot\psi = \frac{\sin\phi}{1-\cos\phi}.$$

At what angle is the curve cut by a line through the cusp perpendicular to the axis?

4. Prove that, for the cardioid,
$$\psi = \frac{\phi}{2}.$$

5. Show that the tangent to the cardioid is parallel to the axis of the curve when $\phi = \frac{2}{3}\pi$.

6. At what points of the cardioid is the tangent perpendicular to the axis of the curve?

7. Determine the rectangle which circumscribes the cardioid and has two of its sides parallel to the axis of the curve.

8. Show that, for the lemniscate,
$$r^2 = a^2 \cos 2\theta,$$
the angle ψ is given by the equation:
$$\cot\psi = -\tan 2\theta.$$
Hence, show that
$$\psi = \frac{\pi}{2} + 2\theta.$$

9. At what points of the lemniscate is the tangent parallel to the axis * of the curve?

Ans. At the point for which $\theta = \pi/6$, and the points which correspond to it by symmetry.

10. The points of the curve
$$r = f(\phi),$$
at which the tangent is parallel to the prime vector, are evidently those for which
$$y = r\sin\phi,$$

* The *axis* of any curve is a line of symmetry. The lemniscate has two such lines. The axis referred to in the text is that one of these lines which passes through the vertices of the curve.

considered as a function of ϕ through the mediation of the equation of the curve, has a maximum, a minimum, or a certain point of inflection. For these points, then,

$$\frac{dy}{d\phi} = r\cos\phi + \sin\phi\,\frac{dr}{d\phi} = 0.$$

Show that this condition is equivalent to the one used above in the special cases considered, namely:

$$\psi + \phi = \pi.$$

11. Plot the curve, $\quad r = a\cos 2\theta,$

taking $a = 5$ cm. Show that for this curve

$$\cot\psi = -2\tan 2\theta.$$

12. At what points of the curve of question 11 is the tangent parallel to the axis?

Ans. For one of the points, $\tan\theta = \frac{1}{\sqrt{5}}$.

13. Plot the curve, $\quad r = a\cos 3\theta,$

taking $a = 5$ cm. Show that

$$\cot\psi = -3\tan 3\theta.$$

14. At what points of the curve of question 13 is the tangent parallel to the axis of the lobe?

Ans. For one of these points, $\tan\theta = \sqrt{1 + \frac{2}{\sqrt{3}}}$.

15. The equation $\quad r = \dfrac{m}{1 + \sin\phi}$

represents a parabola referred to its focus as pole. Give a direct proof that the tangent to this curve at any point bisects the angle formed by the focal radius drawn to this point and a parallel to the axis through the point.

16. Show that the tangent to the hyperbola

$$r = \frac{m}{1 - \sqrt{3}\cos\phi}$$

at the extremity of the latus rectum makes an angle of 60° with the transverse axis.

17. Prove that the tangent to the ellipse

$$r=\frac{\mu}{\sqrt{3}-\cos\phi}$$

at the extremity of the latus rectum makes an angle of 30° with the major axis.

9. Differential of Arc. Let

(1) $$y=f(x)$$

be the equation of a given curve. Let P, with the coordinates (x, y), be a variable point, and A a fixed point of the curve. Denote the length of the arc AP by s. Then s is a function of x; for, when x is given, we know P and thus s.

It is possible to determine the derivative of s, $D_x s$, as follows. By the Pythagorean Theorem we have (Chap. IV, Fig. 33),

$$\overline{PP'}^2=\Delta x^2+\Delta y^2.$$

Hence
$$\left(\frac{\overline{PP'}}{\Delta x}\right)^2=1+\left(\frac{\Delta y}{\Delta x}\right)^2.$$

Let Δx approach 0 as its limit. Then

$$\lim_{\Delta x\doteq 0}\left(\frac{\overline{PP'}}{\Delta x}\right)^2=1+\lim_{\Delta x\doteq 0}\left(\frac{\Delta y}{\Delta x}\right)^2=1+(D_x y)^2.$$

Since by § 3 the chord $\overline{PP'}$ and the arc $\overset{\frown}{PP'}=\Delta s$ are equivalent infinitesimals, it follows from the Fundamental Theorem of Chap. IV, § 2 that, in the above equation, $\overline{PP'}$ can be replaced by Δs. Hence

$$\lim_{\Delta x\doteq 0}\left(\frac{\overline{PP'}}{\Delta x}\right)^2=\lim_{\Delta x\doteq 0}\left(\frac{\Delta s}{\Delta x}\right)^2=(D_x s)^2,$$

and consequently

(2) $$(D_x s)^2=1+(D_x y)^2.$$

On replacing the derivatives in (2) by their values in terms of differentials, we have

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

or

(3) $$ds^2 = dx^2 + dy^2.$$

This formula is easily interpreted geometrically by means of the triangle PMQ, Fig. 33. Since

$$PM = dx \qquad \text{and} \qquad MQ = dy,$$

it follows from the Pythagorean Theorem that

(4) $$PQ = ds.$$

It is obvious geometrically that ds and Δs differ from each other by an infinitesimal of higher order; *i.e.* that they are equivalent infinitesimals.*

Formulas for $\sin\tau$, $\cos\tau$. From the triangle PMQ we can write down two further formulas:

(5) $$\sin\tau = \frac{dy}{ds}, \qquad \cos\tau = \frac{dx}{ds}.$$

These formulas presuppose a suitable choice of τ. As s increases, the point P describes the curve in a definite sense. Let this be chosen as the positive sense of the tangent line at P. Then τ shall be the angle between the positive axis of x and this line. If τ were taken as the angle which the oppositely directed tangent makes with the positive axis of x, the $-$ sign must be written before each right-hand side in (5).

The formulas (5) suggest that x and y can be taken as functions of s:

$$x = \phi(s), \qquad y = \psi(s).$$

* In case the coordinates x and y are expressed as functions of a third variable t, dx will not in general be equal to Δx, but will differ from it by an infinitesimal of higher order. The triangle PMQ will then be replaced by a similar triangle PM_1Q_1, in which M_1 lies on the line PM, its distance from M being an infinitesimal of higher order.

This is, of course, always possible, since, when s is given, P and hence also x and y, are determined.

Since

$$(6) \qquad ds = \pm\sqrt{dx^2 + dy^2},$$

we have from (5)

$$(7) \qquad \sin\tau = \pm\frac{dy}{\sqrt{dx^2+dy^2}}, \qquad \cos\tau = \pm\frac{dx}{\sqrt{dx^2+dy^2}},$$

no matter what choices of s and τ are made.* Furthermore,

$$(8) \qquad \sin\tau = \pm\frac{\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}, \qquad \cos\tau = \pm\frac{1}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}.$$

Which sign is to be used in (8) depends on which of the two possible determinations has been chosen for τ. Thus τ in a given case might be $30°$ or $30° + 180° = 210°$. If the first choice were made, $\tau = 30°$, then $\sin\tau$, $\cos\tau$, and $dy/dx = \tan\tau$ would all be positive quantities, and hence the upper signs must be taken. But if the other choice, $\tau = 210°$, is made, then $\sin\tau$ and $\cos\tau$ are negative, and the lower signs hold.

Example. Consider the parabola

$$y = x^2.$$

Let P be a point of the curve which lies in the first quadrant. Since

$$\tan\tau = \frac{dy}{dx} = 2x$$

is here positive, τ may be taken as an angle of the first quadrant. In that case, formulas (8) give

$$\sin\tau = \frac{2x}{\sqrt{1+4x^2}}, \qquad \cos\tau = \frac{1}{\sqrt{1+4x^2}}.$$

* The signs in (6) and (7) are not necessarily the same; also in (7) and (8)

If P is a point of the curve which lies in the second quadrant, $\tan\tau$ is negative, and τ is an angle of the second or fourth quadrant. If we choose to take τ as an angle of the second quadrant, formulas (8) become

$$\sin\tau = -\frac{2x}{\sqrt{1+4x^2}}, \qquad \cos\tau = -\frac{1}{\sqrt{1+4x^2}}.$$

We may, however, equally well take τ as an angle of the fourth quadrant. Then

$$\sin\tau = \frac{2x}{\sqrt{1+4x^2}}, \qquad \cos\tau = \frac{1}{\sqrt{1+4x^2}}.$$

In each case, one of the numbers, $\sin\tau$ and $\cos\tau$, is positive, the other, negative.

Polar Coordinates. Similar considerations in the case of the curve

$$r = f(\theta)$$

lead to the following formulas; cf. Fig. 45:

$$\overline{PP'}^2 = P'M^2 + MP^2.$$

Hence
$$\lim_{\Delta\theta\doteq 0}\left(\frac{\overline{PP'}}{\Delta\theta}\right)^2 = \lim_{\Delta\theta\doteq 0}\left(\frac{P'M}{\Delta\theta}\right)^2 + \lim_{\Delta\theta\doteq 0}\left(\frac{MP}{\Delta\theta}\right)^2.$$

Now, the chord $\overline{PP'}$ and the arc $\overset{\frown}{PP'} = \Delta s$ are equivalent infinitesimals. Moreover, $P'M$ and Δr are equivalent; and MP and $r_0\Delta\theta$ are equivalent. Hence

$$(D_\theta s)^2 = (D_\theta r)^2 + r_0^2.$$

Dropping the subscript and writing the derivatives in terms of differentials we have, then:

(9)
$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2,$$

or

(10)
$$ds^2 = dr^2 + r^2 d\theta^2.$$

Furthermore,

$$(11) \qquad \sin\psi = \frac{r\,d\theta}{ds}, \qquad \cos\psi = \frac{dr}{ds},$$

the tangent PT being drawn in the direction of the increasing s, and ψ being taken as the angle from the radius vector produced to the positive tangent.

10. Rates and Velocities. The principles of velocities and rates were treated in Chapter III, § 8. We are now in a position to deal with a wider range of problems.

We note the following formulas. Let a point P describe the curve

$$y = f(x).$$

Let s denote the length of the arc, measured from an arbitrary point in an arbitrary sense, and let τ be the angle from the positive direction of the axis of x to the tangent at P drawn in the direction of the increasing arc. Then the components of the velocity ($v = ds/dt$) of P along the axes are, respectively:

$$(1) \qquad \frac{dx}{dt} = v\cos\tau, \qquad \frac{dy}{dt} = v\sin\tau.$$

Let a point P describe the curve

$$(2) \qquad r = F(\theta).$$

Let s denote the length of the arc, measured from an arbitrary point in an arbitrary sense; and let ψ be the angle from the radius vector, produced beyond P, to the tangent at P drawn in the direction of the increasing arc. Then the components of the velocity ($v = ds/dt$) of P along the radius vector produced and perpendicular to the same (the sense of the increasing θ being taken as positive for the latter) are respectively:

$$(3) \qquad \frac{dr}{dt} = v\cos\psi, \qquad r\frac{d\theta}{dt} = v\sin\psi.$$

Example 1. A railroad train is running at the rate of 30 miles an hour along a curve in the form of a parabola:

$$y^2 = 1000x,$$

the axis of the parabola being east and west, and the foot being taken as the unit of length. The sun is just rising in the east. Find how fast the shadow of the locomotive is moving along the wall of the station, which is north and south, when the distance of the shadow from the axis of the parabola is 300 ft.

The first thing to do is to draw a suitable figure, introduce suitable variables, and set down all the data not already put into evidence by the figure. Thus in the present case we have, in addition to the accompanying figure, the further data: (*a*) the velocity of the train; this must be expressed in *feet per second*, since we wish to retain the foot as the unit of length for the equation of the curve. Now, 30 miles an hour is equivalent to 44 feet a second. On the other hand, another expression for the velocity is ds/dt. Hence we have, on equating these two values,

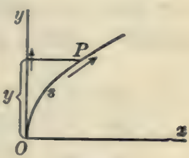


FIG. 47

$$\frac{ds}{dt} = 44.$$

(*b*) We must set down explicitly at this point the equation of the curve,

$$y^2 = 1000x.$$

To sum up, then, we first draw the figure and then write down the supplementary data:

Given   *a*) $\dfrac{ds}{dt} = 44,$

and   *b*) $y^2 = 1000x.$

The second thing to do is to make clear what the problem is. In the present case it can be epitomized as follows:

To find $\left(\dfrac{dy}{dt}\right)_{y=300}.$

We are now ready to consider what methods are at our disposal for solving the problem. We observe that ds occurs in

the data. Obviously, then, we must make use of the one general theorem we know which gives an expression for ds when the equation of the curve comes to us in Cartesian coordinates, — namely, the theorem:

$$ds^2 = dx^2 + dy^2.$$

Since dx occurs neither in the data nor in the conclusion, we wish to eliminate it. This can be done by means of the equation of the path b). Differentiating b) we have:

$$2\,y\,dy = 1000\,dx.$$

Hence
$$dx = \frac{y\,dy}{500}.$$

Consequently
$$ds^2 = \frac{y^2\,dy^2}{500^2} + dy^2$$

and
$$ds = \sqrt{\frac{y^2}{500^2} + 1}\,dy.$$

The next step is obvious; divide through by dt:

$$\frac{ds}{dt} = \sqrt{\frac{y^2}{500^2} + 1}\,\frac{dy}{dt}.$$

In this last equation, replace ds/dt by its known value from a), and we now have an equation for determining dy/dt:

$$\frac{dy}{dt} = \frac{44}{\sqrt{\frac{y^2}{500^2} + 1}}.$$

Finally, bring into action the particular value of y with which alone the proposed equation is concerned, namely, $y = 300$:

$$\left(\frac{dy}{dt}\right)_{y=300} = \frac{44}{\sqrt{.6^2 + 1}} = \frac{44}{\sqrt{1.36}} = 37.73,$$

or, the rate at which the shadow is moving along the wall of the station is 37.73 ft. a second.

Angular Velocity. By the *angular velocity*, ω, with which a line is turning in a given plane is meant the rate at which the angle, ϕ, made by the rotating line with a fixed line, is increasing:

$$\omega = \frac{d\phi}{dt}.$$

Example 2. A point is describing the cardioid

$$r = a(1 - \cos\theta)$$

at the rate of c ft. a second. Find the rate at which the radius vector drawn to the point is turning when $\theta = \pi/2$.

The formulation of this problem is as follows:

Given $\quad a)\quad \dfrac{ds}{dt} = c$

and $\quad b)\quad r = a(1 - \cos\theta).$*

To find $\quad \left(\dfrac{d\theta}{dt}\right)_{\theta=\frac{\pi}{2}}.$

Since, from § 9 (10),

$$ds^2 = dr^2 + r^2\,d\theta^2,$$

and from b),

$$dr = a\sin\theta\,d\theta,$$

it follows that

$$\begin{aligned} ds^2 &= a^2\sin^2\theta\,d\theta^2 + a^2(1-\cos\theta)^2\,d\theta^2 \\ &= a^2\,d\theta^2\left[\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta\right] \\ &= 2\,a^2d\theta^2\cdot(1-\cos\theta) = 4\,a^2\sin^2\frac{\theta}{2}\,d\theta^2. \end{aligned}$$

Hence, s being measured from the cusp,

$$ds = 2\,a\sin\frac{\theta}{2}\,d\theta,$$

and

$$\frac{ds}{dt} = 2\,a\sin\frac{\theta}{2}\frac{d\theta}{dt}.$$

* The student should make a free-hand drawing of the curve.

Consequently, by the aid of a)

$$\frac{d\theta}{dt}=\frac{c}{2\,a\sin\frac{\theta}{2}},$$

and thus, finally
$$\left(\frac{d\theta}{dt}\right)_{\theta=\frac{\pi}{2}}=\frac{c}{a\sqrt{2}}.$$

EXERCISES

1. A point describes a circle of radius 200 ft. at the rate of 20 ft. a second. How fast is its projection on a fixed diameter travelling when the distance of the point from the diameter is 100 ft.? *Ans.* 10 ft. a second.

2. A flywheel 15 ft. in diameter is making 3 revolutions a second. The sun casts horizontal rays which lie in or are parallel to the plane of the flywheel. A small protuberance on the rim of the wheel throws a shadow on a vertical wall. How fast is the shadow moving when it is 4 ft. above the level of the axle?

3. A revolving light sends out a bundle of rays that are approximately parallel, its distance from the shore, which is a straight beach, being half a mile, and it makes one revolution in a minute. Find how fast the light is travelling along the beach when at the distance of a quarter of a mile from the nearest point of the beach.

4. A point moves along the curve $r=1/\theta$ at the rate of 6 ft. a second. How fast is the radius vector turning when $\theta=2\pi$?

5. In the example of the ladder, Chap. III, § 8, Ex. 5, find how fast the ladder is rotating at the instant in question.

6. At what rate is the direction of the second ship from the first changing at the instant in question, in Ex. 2 of Chap. III, § 8?

7. How fast is the direction of the man from the lamp-post changing in Ex. 12 of Chap. III, § 8?

8. The sun is just setting as a baseball is thrown vertically upward so that its shadow mounts to the highest point of the dome of an observatory. The dome is 50 ft. in diameter. Find how fast the shadow of the ball is moving along the dome one second after it begins to fall, and also how fast it is moving just after it begins to fall.

9. Let AB, Fig. 48, represent the rod that connects the piston of a stationary engine with the fly-wheel. If u denotes the velocity of A in its rectilinear path, and v that of B in its circular path, show that

$$u = (\sin\theta + \cos\theta\tan\phi)v.$$

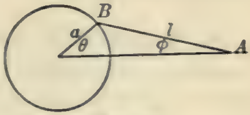


FIG. 48

10. Find the velocity of the piston of a locomotive when the speed of the axle of the drivers is given.

FIG. 49

11. A drawbridge 30 ft. long is being slowly raised by chains passing over a windlass and being drawn in at the rate of 8 ft. a minute. A distant electric light sends out horizontal rays and the bridge thus casts a shadow on a vertical wall, consisting of the other half of the bridge, which has been already raised. Find how fast the shadow is creeping up the wall when half the chain has been drawn in.

12. A man walks across the floor of a semicircular rotunda 100 ft. in diameter, his speed being 4 ft. a second, and his path the radius perpendicular to the diameter joining the extremities of the semicircle. There is a light at one of the latter points. Find how fast the man's shadow is moving along the wall of the rotunda when he is halfway across.

13. A man in a train that is running at full speed looks out of the window in a direction perpendicular to the track. If he fixes his attention successively for short intervals of time on objects at different distances from the train, show that the rate at which he has to turn his eyes to follow a given object is inversely proportional to its distance from him.

14. Water is flowing out of a vessel of the form of an inverted cone, whose semi-vertical angle is 30°, at the rate of a quart in 2 minutes, the opening being at the vertex. How fast is the level of the water falling when there are 4 qt. of water still in?

15. Suppose that the locomotive of the first of the Examples worked in the text is approaching the station at night at the rate of 20 miles an hour, its headlight sending out a bundle of parallel rays. How fast will the spot of light be moving along the wall of the station when the distance of the headlight from the vertex A of the parabola, measured in a straight line, is 500 ft.?

Assume that the wall is perpendicular to the axis of the parabola and distant 75 ft. from the vertex.

16. In the preceding question, how fast will the bundle of rays be rotating?

17. A point describes a circle with constant velocity. Show that the velocity with which its projection moves along a given diameter is proportional to the distance of the point from this diameter.

18. A point P describes the arc of the ellipse

$$9x^2 + 4y^2 = 36,$$

which lies in the first quadrant, at the rate of 12 ft. a second. The tangent at P cuts off a right triangle from the first quadrant. How fast is the area of this triangle changing when P passes through the extremity of the latus rectum? Is the area increasing or decreasing?

19. A point P describes the cardioid

$$r = 5(1 - \cos\theta)$$

at the rate of 12 cm. a second. The tangent at P cuts the axis of the curve in Q. How fast is Q moving when $\theta = \pi/2$?

20. The sun is just setting in the west as a horse is running around an elliptical track at the rate of m miles an hour. The axis of the ellipse lies in the meridian. Find the rate at which the horse's shadow moves on a fence beyond the track and parallel to the axis.

CHAPTER VI

LOGARITHMS AND EXPONENTIALS

1. Logarithms. The logarithms with which the student is familiar are those which are ordinarily used for computation. The base is 10, and the definition of $\log_{10} x$ is as follows:

$$y = \log_{10} x \qquad \text{if} \qquad 10^y = x.$$

These are called *denary*, or *Briggs's*, or *common* logarithms.

More generally, any positive number, a, except unity, can be taken as the base, the definition of $\log_a x$ then being:

$$(1) \qquad y = \log_a x \qquad \text{if} \qquad a^y = x.$$

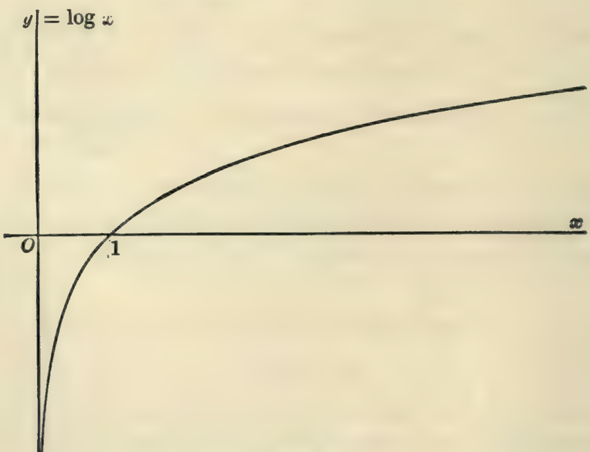


FIG. 50

The accompanying figure represents in character the graph of the function $\log_a x$ for any $a > 1$. It is drawn to scale for

$a = 2.71828$. The reason for this choice of a will appear shortly.

From the definition it follows at once that

$$(2) \qquad \log_a 1 = 0, \qquad \log_a a = 1.$$

Only positive numbers have logarithms. For, a^y is always positive. Hence, if x be given a negative value (or the value 0), the second equation under (1) above cannot be satisfied by any value of y.

The two leading properties of logarithms are expressed by the equations:*

$$(\mathrm{I}) \qquad \log P + \log Q = \log (PQ)$$

$$(\mathrm{II}) \qquad \log P^n = n \log P.$$

Here, P and Q are any two positive numbers whatever, and n is any number, positive, negative, or zero. The base, a, is arbitrary. Thus

$$\log 10 = \log 2 + \log 5$$

and

$$\log \sqrt{7} = \log 7^{\frac{1}{2}} = \tfrac{1}{2} \log 7.$$

From equation (I) it follows that

$$(3) \qquad \log \frac{1}{Q} = -\log Q$$

and

$$(4) \qquad \log \frac{P}{Q} = \log P - \log Q.$$

For, if we set $P = 1/Q$ in (I), we have

$$\log 1 = \log \frac{1}{Q} + \log Q.$$

But, by (2),

$$\log 1 = 0.$$

* The student should recall the proofs of these theorems, which he learned in the earlier study of logarithms, and make sure that he can reproduce them. Proofs of the theorems are given in the author's *Differential and Integral Calculus*, p. 76.

Hence
$$\log\frac{1}{Q}=-\log Q, \qquad \text{q. e. d.}$$

Again, write (1) in the form
$$\log(PQ')=\log P+\log Q',$$
and now set $Q'=1/Q$. Then
$$\log\frac{P}{Q}=\log P+\log\frac{1}{Q}.$$

But
$$\log\frac{1}{Q}=-\log Q.$$

Hence
$$\log\frac{P}{Q}=\log P-\log Q, \qquad \text{q. e. d.}$$

For example,
$$\log(a+b)-\log a=\log\left(1+\frac{b}{a}\right),$$
as we see by setting, in equation (4),
$$P=a+b, \qquad Q=a.$$

As a further example of the application of equation (II) we may cite the following:
$$\frac{\log(a+b)}{h}=\log\{(a+b)^{\frac{1}{h}}\}.$$

For, if $P=a+b$ and $n=\dfrac{1}{h}$, the left-hand side of this equation has the value $n\log P$.

A Further Property of Logarithms. When it is desired to express a logarithm given to a certain base, a, in terms of logarithms taken to a second base, b, the following relation is needed:

$$\text{(III)} \qquad \log_a x=\frac{\log_b x}{\log_b a}.$$

The proof of (III) is as follows. Let
$$y=\log_a x, \qquad a^y=x.$$

Take the logarithm of each side of this equation to the base b:

(5) $$\log_b a^y = \log_b x.$$

But the left-hand side can be transformed by (II), if in (II) we take b as the base, thus having

$$\log_b P^n = n \log_b P.$$

Here, let

$$P = a, \qquad n = y.$$

Then $$\log_b a^y = y \log_b a,$$

and (5) now becomes:

$$y \log_b a = \log_b x.$$

Hence

$$y = \frac{\log_b x}{\log_b a}, \qquad \text{or} \qquad \log_a x = \frac{\log_b x}{\log_b a}, \qquad \text{q. e. d.}$$

Example. Let $b = 10$ and let $a = 2.718$. To find $\log_a 2$. From (III),

$$\log_a 2 = \frac{\log_{10} 2}{\log_{10} 2.718} = \frac{.3010}{.4343} = .6932.$$

Two Identities. Just as, for example,

$$\sqrt[3]{x^3} = x \qquad \text{and} \qquad (\sqrt[3]{x})^3 = x,$$

no matter what value x may have, so we can state two identities for logarithms and exponentials. In the second equation (1), replace y by its value from the first equation. Thus the equation

(6) $$a^{\log_a x} = x$$

is seen to hold for all positive values of x.

Secondly, replace x in the first equation (1) by its value from the second equation:

$$y = \log_a a^y.$$

We can equally well write x instead of y, understanding now by x any number whatever, and we have, then, the

identity

$$\log_a a^x = x. \tag{7}$$

This equation holds for all values of x, positive, negative, or zero.

EXERCISES

1. Show that
$$\log_{10} .8950 = -.0482.$$

2. Find $\log_{10} .09420$. *Ans.* -1.0259.

3. Compute $2.718^{.5642}$. *Ans.* 1.758.

4. Compute $2.718^{-.8710}$. *Ans* 0.4186.

5. Compute π^π. 6. Compute $\sqrt{2}^{\sqrt{3}}$.

7. Show that
$$\log \tan \theta = \log \sin \theta - \log \cos \theta, \qquad 0 < \theta < \frac{\pi}{2}.$$

8. Show that
$$\log \sin \theta + \log \cos \theta = \log \frac{\sin 2\theta}{2}, \qquad 0 < \theta < \frac{\pi}{2}.$$

9. Show that
$$\log \frac{1 - \cos \theta}{2} = 2 \log \sin \frac{\theta}{2}, \qquad 0 < \theta < 2\pi.$$

10. If (x, y) are the Cartesian coordinates of a point distinct from the origin, and (r, θ) the polar coordinates of the same point, show that
$$\log r = \tfrac{1}{2} \log (x^2 + y^2).$$

11. Prove that
$$\log (a^2 - b^2) = \log (a + b) + \log (a - b),$$
provided $a + b$ and $a - b$ are both positive quantities.

12. Simplify the expression
$$\log (1 + x^6) - \log (1 + x^2).$$

13. Show that
$$\sqrt{(e^x - e^{-x})^2 + 4} = e^x + e^{-x},$$
where e has the value 2.7182.

14. Simplify the expression

$$\sqrt{\left(\frac{a^x - a^{-x}}{2}\right)^2 + 1}, \qquad 0 < a.$$

15. Show that

$$\frac{1}{t}\log(1+t) = \log(1+t)^{\frac{1}{t}}.$$

2. Differentiation of Logarithms. In order to differentiate the function

$$y = \log_a x,$$

it is necessary to go back to the definition of a derivative, Chap. II, § 1, and carry through the process step by step.

Give to x an arbitrary positive value, x_0, and compute the corresponding value, y_0, of the function:

$$(1) \qquad y_0 = \log_a x_0.$$

Next, give to x an increment Δx (subject merely to the restriction that $x_0 + \Delta x$ is positive and $\Delta x \neq 0$) and compute the new value, $y_0 + \Delta y$, of the function:

$$(2) \qquad y_0 + \Delta y = \log_a (x_0 + \Delta x).$$

From (1) and (2) it follows that

$$\frac{\Delta y}{\Delta x} = \frac{\log_a (x_0 + \Delta x) - \log_a x_0}{\Delta x}.$$

It is at this point that the specific properties of the logarithmic function come into play for the purpose of transforming the last expression. By § 1, (4),

$$\log_a (x_0 + \Delta x) - \log_a x_0 = \log_a \left(1 + \frac{\Delta x}{x_0}\right),$$

and hence

$$(3) \qquad \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x_0}\right).$$

We next replace the variable Δx by a new variable t as follows:

$$t = \frac{\Delta x}{x_0} \qquad \text{or} \qquad \Delta x = x_0 t.$$

Thus (3) takes on the form

$$\frac{\Delta y}{\Delta x} = \frac{1}{x_0 t} \log_a (1 + t) = \frac{1}{x_0}\left[\frac{1}{t} \log_a (1 + t)\right].$$

From (II), § 1, the bracket is seen to have the value

$$\log_a (1 + t)^{\frac{1}{t}},$$

and hence

(4) $$\frac{\Delta y}{\Delta x} = \frac{1}{x_0} \log_a (1 + t)^{\frac{1}{t}}.$$

As Δx approaches 0 as its limit, t also approaches 0, and so

(5) $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{1}{x_0} \lim_{t \doteq 0} \log_a (1 + t)^{\frac{1}{t}}.$$

Now, the variable $(1 + t)^{\frac{1}{t}}$ approaches a limit when t approaches 0, and this limit is the number which is represented in mathematics by the letter e; cf. § 3. Its value to five places of decimals is

$$e = 2.71828 \cdots;$$

cf. § 3. Moreover, $\log x$ is a continuous function of x, as is shown in a detailed study of this function.* Hence

$$\lim_{t \doteq 0} \log_a (1 + t)^{\frac{1}{t}} = \log_a \left\{ \lim_{t \doteq 0} (1 + t)^{\frac{1}{t}} \right\} = \log_a e.$$

On substituting this value in the right-hand side of (5) we have:

$$D_x y = \frac{\log_a e}{x_0},$$

* Such a treatment is too advanced to be pursued with profit at this stage. Cf. the author's *Differential and Integral Calculus*, Appendix, p. 417

or, on dropping the subscript:

$$(6) \qquad D_x \log_a x = \frac{\log_a e}{x}.$$

Thus if the usual base, $a = 10$, be taken, the formula becomes:

$$(7) \qquad D_x \log_{10} x = \frac{.4343 \cdots}{x}.$$

Discussion of the Result. We have met a similar situation before, in the differentiation of the sine. There, if angles be measured in degrees, the fundamental formula reads:

$$D_x \sin x = \frac{\pi}{180} \cos x.$$

In order to get rid of this inconvenient multiplier, we changed the unit of angle from the degree to the radian, and then the formula became:

$$D_x \sin x = \cos x.$$

In the present case, it is possible to do a similar thing. The base, a, is wholly in our control, to choose as we like. Now, for any base, the logarithm of the base is unity, § 1, (2):

$$\log_a a = 1.$$

If, then, we choose as our base the number e:

$$a = e = 2.71828 \cdots$$

the multiplier becomes

$$(8) \qquad \log_a e = \log_e e = 1.$$

For this reason, e is taken as the base of the logarithms used in the Calculus.* These are called NATURAL *logarithms.* They are also called *hyperbolic*, or *Naperian* logarithms, — the latter name after Napier, the inventor of logarithms. But

* The notation e for this number is due to Euler, 1728.

Napier* was the very man who introduced denary logarithms into mathematics, and so the use of his name in connection with natural logarithms is misleading.

Since natural logarithms are always meant in the formulas of the calculus, unless the contrary is explicitly stated, it is customary to drop the index e from the notation $\log_e x$ and to write

$$(9) \qquad y = \log x, \qquad \text{if} \qquad e^y = x.$$

The identities (6) and (7) of § 1 now take on the form:

$$(10) \qquad e^{\log x} = x,$$

$$(11) \qquad \log e^x = x.$$

The formula of differentiation becomes:

$$(12) \qquad D_x \log x = \frac{1}{x}.$$

In differential form it reads:

$$(13) \qquad \frac{d}{dx} \log x = \frac{1}{x},$$

$$(14) \qquad d \log x = \frac{dx}{x}.$$

Example. Differentiate the function

$$u = \log \sin x.$$

Let

$$y = \sin x.$$

Then

$$u = \log y,$$

$$du = d \log y = \frac{dy}{y}, \qquad dy = \cos x\, dx,$$

and

$$du = \frac{\cos x\, dx}{\sin x} = \cot x\, dx.$$

* Napier was a Scotchman, and his discovery was published in 1614.

Hence $$d \log \sin x = \cot x \, dx,$$

or $$\frac{d}{dx} \log \sin x = \cot x.$$

EXERCISES

Differentiate the following functions.

1. $u = \log \cos x.$ $\qquad \dfrac{du}{dx} = -\tan x.$

2. $u = \log \tan x.$ $\qquad \dfrac{du}{dx} = \cot x + \tan x$

3. $u = \log \cot x.$ $\qquad \dfrac{du}{dx} = \dfrac{-2}{\sin 2x}.$

4. $u = \log \sec x.$ $\qquad$ 5. $u = \log \csc x.$

6. $u = \log \dfrac{x}{1-x}.$ $\qquad \dfrac{du}{dx} = \dfrac{1}{x} + \dfrac{1}{1-x}.$

7. $u = \log \dfrac{a+x}{a-x}.$ $\qquad \dfrac{du}{dx} = \dfrac{2a}{a^2 - x^2}.$

8. $u = \log \sqrt{a^2 + x^2}.$ $\qquad \dfrac{du}{dx} = \dfrac{x}{a^2 + x^2}.$

9. $u = \log (1 - \cos x).$ $\qquad \dfrac{du}{dx} = \cot \dfrac{x}{2}.$

10. $u = \log (1 + \cos x).$ $\qquad \dfrac{du}{dx} = -\tan \dfrac{x}{2}.$

3. The Limit $\lim\limits_{t=0} (1+t)^{\frac{1}{t}}$. Since this limit is fundamental in the differentiation of the logarithm, a detailed discussion of it is essential to completeness. Let us set

(1) $$s = (1+t)^{\frac{1}{t}}$$

and compute the value of s for values of t near 0. Suppose $t = .1$. Then

$$s = (1.1)^{10},$$

and this number is found by the usual processes with logarithms to be 2.59.

Further pairs of corresponding values (t, s) are found in a similar manner. In particular, the student can verify the correctness of the following table of values:*

t	-0.1	$-.01$	$-.001$	$\ldots$	$+.001$	$+.01$	$+0.1$
s	2.87	2.73	2.72	$\ldots$	2.72	2.70	2.59

The foregoing table indicates strongly that, when t approaches the limit 0 from either side, the variable s is approaching a limit whose value, to three significant figures, is 2.72. This is in fact the case.† The exact value of the limit is denoted by the letter e:

$$\lim_{t \doteq 0} (1 + t)^{\frac{1}{t}} = e = 2.71828 \cdots. \tag{2}$$

4. The Compound Interest Law. The limit (2) of § 3 presents itself in a variety of problems, typical for which is that of finding how much interest a given sum of money would bear if the interest were compounded continuously, so that there is no loss whatever. For example, \$1000, put at interest at 6 %, amounts in a year to \$1060, if the interest is not compounded at all. If it is compounded every six months, we have

$$\$\,1000\left(1 + \frac{.06}{2}\right)$$

as the amount at the end of the first six months, and this must be multiplied by $\left(1 + \frac{.06}{2}\right)$ to yield the amount at the end of the second six months, the final amount thus being

$$\$\,1000\left(1 + \frac{.06}{2}\right)^2.$$

* To compute the middle entries in this table a six-place table of logarithms is needed.

† For a rigorous proof cf. the author's *Differential and Integral Calculus*, p. 79.

It is readily seen that if the interest is compounded n times in a year, the principal and interest at the end of the year will amount to

$$1000\left(1+\frac{.06}{n}\right)^n$$

dollars, and we wish to find the limit of this expression when $n=\infty$. To do so, write it in the form:

$$1000\left[\left(1+\frac{.06}{n}\right)^{\frac{n}{.06}}\right]^{.06}$$

and set $t=\frac{.06}{n}$. The bracket thus becomes

$$(1+t)^{\frac{1}{t}},$$

and its limit is e. Hence the desired result is

$$1000\,e^{.06}=1061.84.*$$

EXERCISE

If \$1000 is put at interest at 4 %, compare the amounts of principal and interest at the end of 10 years, (*a*) when the interest is compounded semiannually, and (*b*) when it is compounded continuously. *Ans.* A difference of \$5.88.

5. Differentiation of e^x. Before beginning this paragraph the student will turn to Chap. VIII and study carefully § 1.

Since

$$(1) \qquad y=e^x \qquad \text{and} \qquad x=\log y$$

are equivalent equations, the former function can be differentiated by taking the differential of each side of the latter equation:

$$dx=d\log y=\frac{dy}{y}.$$

* The actual computation here is expeditiously done by means of series; see the chapter on Taylor's Theorem.

Hence
$$\frac{dy}{dx} = y,$$
or

(2) $$\frac{de^x}{dx} = e^x,$$

(3) $$de^x = e^x\,dx.$$

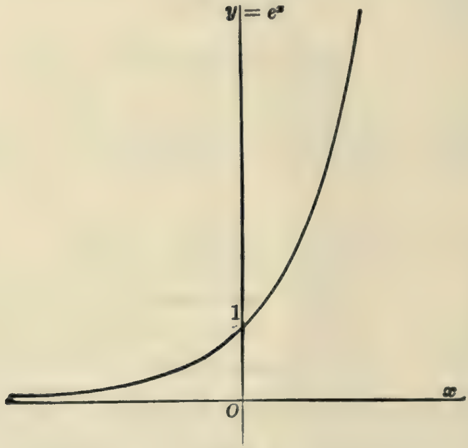


FIG. 51

The function

(4) $$y = a^x$$

could be differentiated in a similar manner. It is, however, simpler to take the logarithm of each side of (4) and then differentiate the new equation:

$$\log y = \log a^x = x \log a,$$

$$d \log y = \frac{dy}{y} = dx \log a.$$

(5) $$da^x = a^x \log a\,dx.$$

Differentiation of x^n. It is now possible to complete the differentiation of this function for the case that n is irrational.

Since by § 2, (10), $x = e^{\log x}$,

it follows that $x^n = e^{n \log x}$,

and hence

$$
\begin{aligned}
d x^n &= d e^{n \log x} \\
&= e^{n \log x} d(n \log x) \\
&= e^{n \log x} \frac{n\,dx}{x} \\
&= x^n \frac{n\,dx}{x}.
\end{aligned}
$$

Thus finally,

(6) $$d x^n = n x^{n-1} dx,$$

no matter what value n may have, provided merely that n is a constant.

Differentiation of $f(x)^{\phi(x)}$. Let it be required, for example, to differentiate the function

$$y = x^x.$$

Here, both base and exponent are variable. Begin by taking the logarithm of each side of the equation:

$$\log y = \log x^x = x \log x.$$

Hence $$d \log y = d(x \log x),$$

or $$\frac{dy}{y} = (1 + \log x)\,dx,$$

and so, finally, $$dy = y(1 + \log x)\,dx$$

or $$d x^x = x^x (1 + \log x)\,dx.$$

The general case, $y = f(x)^{(\phi x)}$,

can be treated in a similar manner.

6. Graph of the Function x^n. For positive values of n the curves

$$y = x^n$$

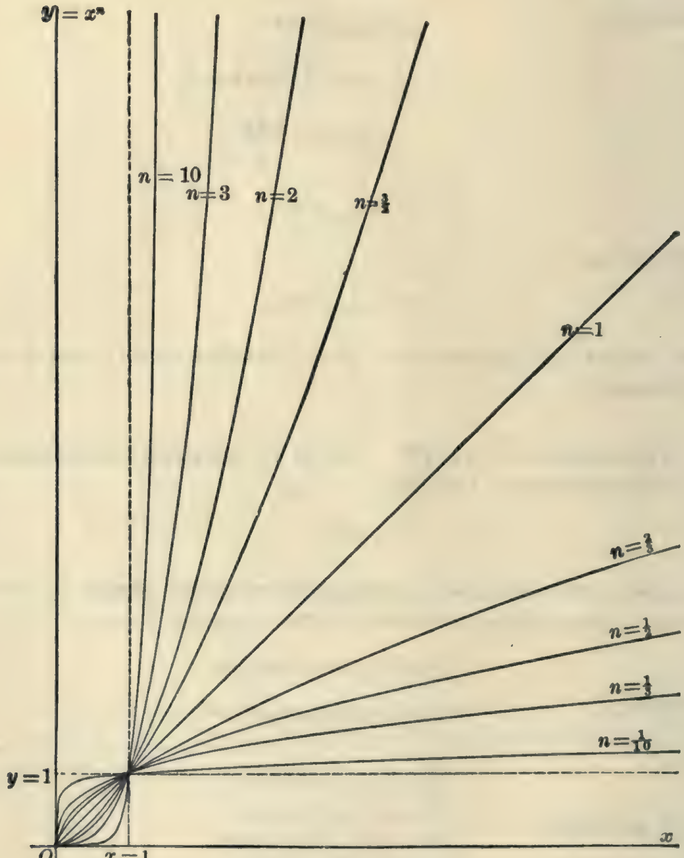


FIG. 52

lie as indicated in the figure. When $n = 1$, we have the ray from the origin, which bisects the angle between the positive axes of x and y.

When $n > 1$, the curve is always concave upward; when $n < 1$, it is concave downward.

All the curves start at the origin and pass through the point (1, 1).

For values of $x > 1$, the larger n, the higher the curve lies. For values of $x < 1$, the reverse is the case.

Let x have any fixed value greater than unity: $x = x' > 1$. Consider the ordinate

$$y = x'^n.$$

As n increases, x'^n increases continuously. This property is the basis of the property of logarithms included in the word *continuous*.

For proofs of the foregoing statements cf. the author's *Differential and Integral Calculus*, p. 27 and Appendix, p. 417.

7. The Formulas of Differentiation to Date. The student will now bring his card of formulas up to date by supplementing it so that it will read as follows:

GENERAL FORMULAS OF DIFFERENTIATION

I. $$d\,cu = c\,du.$$

II. $$d(u+v) = du + dv.$$

III. $$d(uv) = u\,dv + v\,du.$$

IV. $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

SPECIAL FORMULAS OF DIFFERENTIATION

1) $$dc = 0.$$

2) $$dx^n = nx^{n-1}\,dx.$$

3) $$d\sin x = \cos x\,dx.$$

4) $$d\cos x = -\sin x\,dx.$$

5) $$d\tan x = \sec^2 x\,dx.$$

6) $$d \cot x = -\csc^2 x \, dx.$$

7) $$d \log x = \frac{dx}{x}.$$

8) $$d e^x = e^x \, dx.$$

9) $$d a^x = a^x \log a \, dx.$$

To obtain facility in the use of the new results it is desirable that the student work a good number of simple exercises.

Example 1. To differentiate the function

$$u = e^{ax}.$$

Let $$y = ax.$$

Then $$u = e^y,$$

$$du = de^y = e^y \, dy = e^{ax}(a \, dx).$$

Hence

$$d e^{ax} = a e^{ax} \, dx \qquad \text{or} \qquad \frac{d}{dx} e^{ax} = a e^{ax}.$$

Example 2. If

$$u = A \cos (nt + \gamma),$$

show that *

$$\frac{d^2u}{dt^2} + n^2 u = 0.$$

To do this, compute first $\frac{du}{dt}$. The computation is readily effected by taking the differential of each side of the given equation:

$$\begin{aligned} du &= A \, d \cos (nt + \gamma) \\ &= A[-\sin (nt + \gamma) \, d(nt + \gamma)] \\ &= -An \sin (nt + \gamma) \, dt, \end{aligned}$$

* Such an equation as the following is called a *differential equation*, and any function which, when substituted for u, satisfies the equation is called a *solution*.

$$\frac{du}{dt} = -An\sin(nt+\gamma).$$

Next, compute $\frac{d^2u}{dt^2}$. Since

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d\left(\frac{du}{dt}\right)}{dt},$$

we take the differential of each side of the equation for $\frac{du}{dt}$:

$$\begin{aligned} d\left(\frac{du}{dt}\right) &= -An\,d\sin(nt+\gamma) \\ &= -An[\cos(nt+\gamma)d(nt+\gamma)] \\ &= -An^2\cos(nt+\gamma)\,dt. \end{aligned}$$

Hence, on dividing through by dt, we have:

$$\frac{d^2u}{dt^2} = -An^2\cos(nt+\gamma).$$

If now we multiply the given value of u by n^2 and add the product to the value just obtained for $\frac{d^2u}{dt^2}$, the result is identically 0, *i.e.* 0 for all values of t:

$$\frac{d^2u}{dt^2} + n^2u = 0, \qquad \text{q. e. d.}$$

EXERCISES

Differentiate the following functions.

1. $u = e^{-x^2}$. $\qquad \frac{du}{dx} = -2xe^{-x^2}$.

2. $u = e^{\sin x}$. $\qquad \frac{du}{dx} = e^{\sin x}\cos x$.

3. $u = (e^x + e^{-x})^2$. $\qquad \frac{du}{dx} = 2(e^{2x} - e^{-2x})$.

4. $u = 10^x$. $\qquad \dfrac{du}{dx} = (2.30259 \cdots)10^x$.

5. $u = x^{10}\, 10^x$. $\qquad \dfrac{du}{dx} = x^9\, 10^x(10 + 2.30259x)$.

6. $u = \log(\sec x + \tan x)$. $\qquad \dfrac{du}{dx} = \sec x$.

7. $u = x^2 \log x$. $\qquad \dfrac{du}{dx} = x(1 + 2\log x)$.

8. $u = x^3 \log(a - x)$.

9. $u = e^{-x} \log(2x + 3)$.

10. $u = e^{-at} \cos(nt - \gamma)$.

11. $u = e^{-\kappa t}(A \cos nt + B \sin nt)$.

12. $u = \dfrac{x \log x}{x + 1} - \log(x + 1)$. $\qquad \dfrac{du}{dx} = \dfrac{\log x}{(x + 1)^2}$.

13. $u = \log(x + \sqrt{x^2 - a^2})$. $\qquad \dfrac{du}{dx} = \dfrac{1}{\sqrt{x^2 - a^2}}$.

14. $u = \log(x + \sqrt{a^2 + x^2})$. $\qquad \dfrac{du}{dx} = \dfrac{1}{\sqrt{a^2 + x^2}}$.

15. $u = \log(e^x + e^{-x})$.

16. $u = \dfrac{\sin x + \cos x}{e^x}$.

17. $u = \log \tan \dfrac{x}{2}$. $\qquad \dfrac{du}{dx} = \csc x$.

18. $u = \log \tan \left(\dfrac{\theta}{2} + \dfrac{\pi}{4}\right)$. $\qquad \dfrac{du}{d\theta} = \sec \theta$.

19. $u = \cot \left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)$. $\qquad \dfrac{du}{dx} = \dfrac{1}{1 - \sin x}$.

20. $u = \tan \left(\dfrac{x}{2} - \dfrac{\pi}{4}\right)$. $\qquad \dfrac{du}{dx} = \dfrac{1}{1 + \sin x}$.

21. $u = \log \sqrt{1 + \sin \theta}$.

22. $u = \log \dfrac{\sin x}{x}$.

23. $u = \log \sqrt{1 - \cos x}$.

24. $u = \sqrt{e^x}$.

25. $u = (10^{1+t})^2$.

26. $u = \sqrt[3]{a^{-x}}$.

27. $u = \left(\dfrac{a}{a + b}\right)^x$.

28. $u = \sqrt{10^x}$.

29. $u = x^{\sin x}$. $\qquad \dfrac{du}{dx} = x^{\sin x - 1}(\sin x + x \cos x \log x)$.

30. $u = (\sin x)^{\cos x}$. $\qquad \dfrac{du}{dx} = (\sin x)^{\cos x - 1}(\cos^2 x - \sin^2 x \log \sin x)$

31. $u = x^{\frac{1}{x}}$. $\qquad$ **32.** $u = (\cos x)^{\sin x}$. $\qquad$ **33.** $u = (\tan x)^x$.

34. $u = (\log x^2)^x$. $\qquad$ **35.** $u = (1 + a)^{\frac{x}{a}}$. $\qquad$ **36.** $u = (x^2)^{2x}$.

37 If $u = A \cos nt + B \sin nt$, show that

$$\frac{d^2u}{dt^2} + n^2u = 0.$$

38. If $u = Ce^{-\kappa t} \cos(\sqrt{n^2 - \kappa^2}\, t + \gamma)$, show that

$$\frac{d^2u}{dt^2} + 2\kappa \frac{du}{dt} + n^2u = 0.$$

39. The curve

$$r = ae^{\lambda\theta}$$

is known as the *equiangular spiral*, because the radius vector drawn to any point, P, of the curve, and the tangent at P, always make the same angle with each other. Prove this property, and show that, if the angle from the radius vector produced to the tangent be denoted by α, then

$$\cot \alpha = \lambda.$$

CHAPTER VII

APPLICATIONS

1. The Problem of Numerical Computation. It often happens in practice that we wish to solve a numerical equation in one unknown quantity, or a pair of simultaneous equations in two unknowns, to which the standard methods with which we are familiar do not apply; for example,

$$\cos x = x,$$

or

$$\begin{cases} 2\cot\theta + 2 = \cot\phi, \\ 2\cos\theta + \cos\phi = 2. \end{cases}$$

Such equations usually come to us from physical problems, and the solution is required only to a limited degree of accuracy, — say, to two, three, or possibly four significant figures. Any method, therefore, which yields an approximate solution correct to the prescribed degree of accuracy furnishes a solution of the problem.

In particular, the problem of the determination of the error in the result due to errors in the observations comes under this head.

2. Solution of Equations. Known Graphs.

Example 1. Let it be required to solve the equation

1) $$\cos x = x.$$

We can evidently replace this problem by the following: To find the abscissa of the point of intersection of the curves

2) $$y = \cos x, \qquad y = x.$$

The first of these curves we have plotted accurately to scale. The second is the right line through the origin, which bisects the angle between the positive coordinate axes. It is, therefore, sufficient to lay down a ruler on the graph of the former curve, so that its edge lies along the right line in question, and observe where this line cuts the curve. The result lies between

$$x = .7 \qquad \text{and} \qquad x = .8,$$

and may fairly be taken as $x = .75$. It is understood, as usual in approximate values, that the last figure tabulated does not claim complete accuracy; but we are entitled to a somewhat better result than would be given by the first figure alone.

Example 2. To solve the equation

3) $$x^3 + 2x - 2 = 0.$$

Suppose we have plotted the curve

4) $$y = x^3$$

accurately from a table of cubes. Then the problem can conveniently be formulated as follows:

To find the abscissa of the point of intersection of the curves

5) $$y = x^3 \qquad \text{and} \qquad y = 2 - 2x.$$

The details are left to the student.

Example 3. To find the positive root of the equation

6) $$e^{-\frac{1}{2}x} + 2.92x = 2.14.$$

Here, we can connect up with the graph of the function e^x by making a simple transformation. Let

7) $$x' = -\tfrac{1}{2}x; \qquad x = -2x'.$$

The equation then becomes

8) $$e^{x'} - 5.84x' = 2.14,$$

and we seek to determine the abscissa of that point of intersection of the curves (for simplicity, we drop the accent)

9) $\qquad y = e^x \qquad$ and $\qquad y = 5.84x + 2.14$

which lies to the left of the origin. The second place of decimals in the coefficients is not to be taken too seriously; we make as accurate a drawing as the graph and a well-sharpened pencil permit. Having thus determined the negative x' from the graphs of 9), we find the desired positive x by substituting this value in equations 7). The execution of the details is left to the student.

Example 4. Solve the equation

$$e^x = \tan x, \qquad 0 < x < \frac{\pi}{2}.$$

If one of the curves

$$y = e^x \qquad \text{or} \qquad y = \tan x$$

were plotted on transparent paper, or celluloid, it could be laid down on the other with the axes coinciding and the intersection read off. The same result can be attained by holding the actual graphs up in front of a bright light.

In cases as simple as this, however, free-hand graphs will often yield a good first approximation, and further approximations can be secured by the numerical methods of the later paragraphs.

EXERCISES *

1. Solve the equation

$$\cos x = 2x.$$

2. Find the root of the equation

$$3 \sin x = 2x$$

which lies between 0 and π.

* In solving these exercises only so great accuracy is expected as can be attained from well-drawn graphs of the standard curves. It will be shown in later paragraphs how the solutions can be improved analytically and carried to any desired degree of accuracy.

3. Solve: $$x + \tan x = 1, \qquad 0 < x < \frac{\pi}{2}.$$

4. Solve: $$3 \cos x - 5x = 6, \qquad -\frac{\pi}{2} < x < 0.$$

5. Find the root of the equation
$$\log x^2 + 2 = x$$
which lies between 0 and 1.

6. Solve: $$\sin 2x = x.$$

7. Find all the roots of the equation
$$12x^3 + 4x + 3 = 0.$$

8. The same for $$6x^3 - 5x - 1 = 0.$$

9. The same for $$x^3 - x - 1 = 0.$$

Solve the following equations:

10. $$\cos^3 \theta + .47 \cos \theta - 1.23 = 0, \qquad 0 < \theta < 90°.$$

11. $\sin x = \sqrt{1 - x^2}$. 12. $x^2 + \cos^2 x = 4$.

13. Show that the equation
$$\tan x = x$$
has an infinite number of roots. These can be written in the form
$$x_n = n\pi + \epsilon_n,$$
where ϵ_n is numerically small when n is numerically large.

14. Find the largest value of P for which the equation
$$\cos x + Px = 1$$
admits a solution in the interval $0 < x < \pi$.

15. Find the point of the parabola
$$2y = x^2$$
which is nearest to the point (2, 0).

16. Find the radius of the circle whose center is at (0, 2) and which is tangent to the parabola

$$y^2 = x.$$

3. Interpolation. Consider the equation

$$f(x) = 0. \tag{1}$$

Suppose a root has been located with some degree of accuracy. More precisely, suppose that

$$f(x_1) \qquad \text{and} \qquad f(x_2)$$

are of opposite signs. If the function $f(x)$ is continuous in the interval $x_1 \leqq x \leqq x_2$ and if its derivative is always positive (or always negative) in this interval, then the function is always increasing (or always decreasing) and so must have just one root between x_1 and x_2.

The root can be found approximately as follows. Consider the graph of the function

$$y = f(x). \tag{2}$$

Let $\qquad y_1 = f(x_1), \qquad y_2 = f(x_2),$

and draw the chord through the points (x_1, y_1) and (x_2, y_2). The point in which this chord cuts the axis of x will obviously yield a further approximation to the root sought. Denote this last value by X.

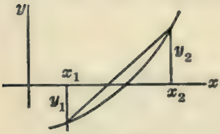


Fig. 53

The equation of the chord is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}. \tag{3}$$

On setting $y = 0$ and solving for x, we have, as the value of X, the following:

$$X = x_1 - \frac{x_2 - x_1}{y_2 - y_1} y_1, \tag{4}$$

or

$$X = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1). \tag{5}$$

We have explained the method in detail and developed, in equations 4) and 5), the analytic formula for the determination of the new approximation, X. In practice, however, it is usually simpler to draw the straight lines of Fig. 53 accurately on a generous scale and read off from the figure the value of X

Example. Consider equation 3) of § 2, Ex. 2:

6) $$x^3+2x-2=0.$$

The curve in question is here

7) $$y=x^3+2x-2,$$

and the graphical solution of § 2 shows that the root is about $x=.7$ or $.8$.

Let $x=x_1=.7$; then y_1 is found to have the value

$$y_1=-.257.$$

Next, let $x=x_2=.8$; then $y_2=.112$.

We have, then, to lay a secant through the points

$$(x_1, y_1)=(.7, -.257) \qquad \text{and} \qquad (x_2, y_2)=(.8, .112).$$

Its equation is given by 3)*:

$$\frac{x-.7}{.8-.7}=\frac{y+.257}{.112+.257}.$$

On setting $y=0$ in this equation and solving for x, we get, cf. 4):

$$X=.7+\frac{.0257}{.369}=.7693.$$

In order to see about how close this approximation is, compute the corresponding value of y:

$$y\,\big|_{x=.7693}=-.0063.$$

We get, then, about two places of decimals, $x=.77$.

* It is desirable that the student should make this determination graphically, as indicated above in the text. He should take 10 cm. to represent the interval of length .1, from $x_1=.7$ to $x_2=.8$.

It is possible to apply the method again, taking now

$$(x_1, y_1) = (.7693, -.0063)$$

and (x_2, y_2) as before. We leave this as an exercise to the student. He should make both the graphical determination with an enlarged scale and the analytic determination of formula 4).

The Method; Not, the Formula. The student may be tempted to use the formula 4) or 5), rather than to go back to the method by which it was derived. This would be unfortunate, for the formula is not easily remembered, whereas the method, once appreciated, can never be forgotten. If the student finds himself in a lumber camp with nothing but the ordinary tables at hand, he may solve his equation if he has once laid hold of the method. It is true that the best way is for him to treat first the literal case and deduce the formula. But this he may not be able to do if he has relied on the formula in the book.

EXERCISES

Apply the method to a good number of the problems at the end of § 2.

4. Newton's Method. Suppose again that it is a question of solving the equation

1) $$f(x) = 0,$$

and suppose we have already succeeded in finding a fairly good approximation, $x = x_1$.

Consider the graph of the function

2) $$y = f(x).$$

Compute $y_1 = f(x_1)$. To improve the approximation, draw the tangent at the point (x_1, y_1). Its equation is:

3) $$y - y_1 = \left(\frac{dy}{dx}\right)_{x=x_1}(x - x_1).$$

Evidently, this line will cut the axis of x at a point very near the point in which the curve 2) cuts this axis. If, then, we set $y=0$ in 3) and solve for x, we shall obtain a second approximation to the root of 1) which we seek. The value of this root will be

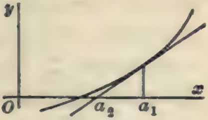


FIG. 54

4)
$$X = x_1 - \frac{y_1}{\left(\dfrac{dy}{dx}\right)_{x=x_1}}.$$

Example 1. Let us apply the method to the Example studied in § 3. In order, however, to have simpler numbers to work with, take $x_1 = .77$ and compute the corresponding y_1; it is found to be: $y_1 = -.0035$.

$$(x_1, y_1) = (.77, -.0035).$$

We must next compute dy/dx from the equation

$$y = x^3 + 2x - 2;$$

$$\frac{dy}{dx} = 3x^2 + 2, \qquad \left(\frac{dy}{dx}\right)_{x=.77} = 3.779.$$

On substituting these values in 3), we have:

$$y + .0035 = 3.779(x - .77).$$

Now set $y=0$ and solve. The result is that given by 4):

$$x = .77 + \frac{.0035}{3.779} = .7709.$$

We have tabulated four figures in the result because this is about the degree of accuracy that seems likely. To test this point, compute y for the value of x which has been found:

$$y\,|_{x=.7709} = -.0001.$$

Since the slope of the graph is greater than unity, the error in x is less than one unit in the fourth place. It is easy to verify the result by computing y for the next larger four-place value of x:

$$y\,|_{x=.7710} = +.0003.$$

Thus we have a complete proof that the root lies between .7709 and .7710, and we see that it lies about one quarter of the way from the first to the second value.

Example 2. It is shown that the equation of the curve in which a chain hangs, — the *Catenary*, — is

5) $$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

where a is a constant. The length of the arc, measured from the vertex, is

6) $$s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right).$$

Let it be required to compute the dip in a chain 32 feet long, its ends being supported at the same level, 30 feet apart.

We can determine the dip from 5) if we know a, and we can get the value of a from 6) by setting $s = 16$, $x = 15$:

$$16 = \frac{a}{2}\left(e^{\frac{15}{a}} - e^{-\frac{15}{a}}\right).$$

Let $x = \dfrac{15}{a}$. Then

$$f(x) = e^x - e^{-x} - \tfrac{32}{15}x = 0,$$

and we wish to know where the curve

7) $$y = f(x) = e^x - e^{-x} - \tfrac{32}{15}x$$

crosses the axis of x.

This curve starts from the origin and, since

$$\frac{dy}{dx} = f'(x) = e^x + e^{-x} - \tfrac{32}{15}$$

is negative for small values of x, the curve enters the fourth quadrant. Moreover,

$$\frac{d^2y}{dx^2} = e^x - e^{-x} > 0, \qquad x > 0,$$

and hence the graph is always concave upward. Finally,

$$f(1) = e - e^{-1} - 2\tfrac{2}{15} = .217 > 0,$$

and so the equation has one and only one positive root, and this root lies between 0 and 1.

It will probably be better to locate the root with somewhat greater accuracy before beginning to apply the above method. Let us compute, therefore, $f(\frac{1}{2})$. By the aid of Peirce's Tables we find:

$$f(.5) = 1.6487 - .6065 - 1.0667 = -.0245 < 0.$$

Comparing these two values of the function:

$$f(.5) = -.02, \qquad f(1) = .22,$$

and remembering that the curve is concave upward, so that the root is somewhat larger than the value obtained by direct interpolation (this value corresponding to the intersection of the chord with the axis of x) we are led to choose as our first approximation $x_1 = .6$:

$$f(.6) = 1.8221 - .5488 - 1.2800 = -.0067,$$

$$f'(.6) = 1.8221 + .5488 - 2.1333 = .2376.$$

Hence the value of the next approximation is

$$X = .6 - \frac{-.0067}{.2376} = .6 + .0282 = .628.$$

To get the next approximation we compute

$$f(.628) = 1.8739 - .5337 - 1.3397 = .0005.$$

Hence the value of the root to three significant figures is .628 with a possible error of a unit or two in the last place, and the value of a we set out to compute is, therefore, $15/.628 = 23.9$.

Remark. Newton's method, like the other methods of this chapter, has the advantage that an error in computing the new approximation will not be propagated in later computations. Such an error will in general hinder us, because we are not

likely to get so good an approximation. But the one test for the accuracy of the approximation is the accurate computation of the corresponding y, and if this is done right, we see precisely how close we are to the desired root.

The function $f(x)$ is usually simple, and it is easy to see whether the curve is concave upward or concave downward near the point where it crosses the axis. We thus have a means of improving the approximation at the same time that we simplify the new value of x. For, if the curve lies to the right of its chord, the approximation by interpolation will be too small; and if the curve lies to the right of its tangent between the point of tangency and the axis of x, the approximation given by Newton's method will also be too small.

Comparison of the Two Methods. When looked at from their geometric side the two methods appear much alike, the first seeming somewhat simpler, since it does not involve the use of derivatives. Why bother, then, with Newton's method? It is not a theoretical question, but purely one of convenience in carrying out the numerical work. It will be found that, as a rule, the first method is preferable in the early stages (usually, merely in the first stage). When, however, a fairly good approximation has been reached, the numerical work involved in Newton's method is generally shorter than that required by interpolation.

EXERCISES

Apply the method to the Exercises of § 2. When, however, the approximation given by the graphical method of § 1 is crude, the method of interpolation may be used to improve it.

5. Direct Use of the Tables.

Example 1. Let us recur to the first example studied, Ex. 1, § 2:

1) $$\cos x = x.$$

The graphical solution gave $x = .75$. Turn now to a table of natural cosines in radian measure, preferably Peirce's Tables. As we run down the table, we find the entries:

RADIANS	COS NAT
.7389	.7392
.7418	.7373

Thus x is seen to lie between .7389 and .7418. It is an excellent exercise for the student to work out the interpolation for himself before we take it up at the end of the paragraph. The answer is: $x = .7391$.

Example 2. Consider the equation

2) $$\tan x = e^x,$$

the desired root lying between 0 and $\pi/2$.

A free-hand drawing of the graphs of the functions

$$y = \tan x, \qquad y = e^x$$

shows that x lies between 1 and 1.5. So the next step is taken conveniently by opening Peirce's Tables to the Trigonometric Functions and Huntington's to the Exponentials, and writing down the two pairs of values of the functions which came nearest together:

x	$\tan x$	e^x
1.3	3.60	3.67
1.4	5.80	4.06

Thus the root is seen to lie between 1.3 and 1.4.

The general case which the above examples are intended to illustrate is the following: — To solve the equation

$$f(x) = \phi(x),$$

where $f(x)$ and $\phi(x)$ are tabulated functions, or functions readily computed.

When the solution has progressed to the point indicated by the examples, the next step can be taken by interpolation, or by Newton's method, as will now be explained.

Interpolation. When two values of the independent variable near together, x_1 and x_2, have been found such that $f(x)$ is greater than $\phi(x)$ for one of them and less than $\phi(x)$ for the other, the best approximation to take next is the one given by the abscissa of the point of intersection of the chords of the graphs of the functions,

$$y = f(x), \qquad y = \phi(x).$$

This value, X, can be found as follows.

Suppose that

$$f(x_1) < \phi(x_1) \qquad \text{and} \qquad f(x_2) > \phi(x_2).$$

Introduce the following notation:

$$\phi(x_1) - f(x_1) = \Delta_1, \qquad f(x_2) - \phi(x_2) = \Delta_2,$$

$$x_2 - x_1 = \delta, \qquad X - x_1 = h.$$

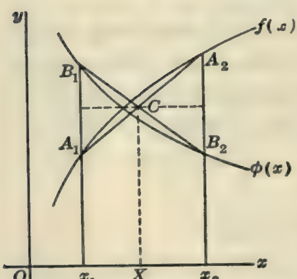


FIG. 55

From the figure, the triangles A_1CB_1 and A_2CB_2 are similar, and

$$A_1B_1 = \Delta_1, \qquad A_2B_2 = \Delta_2.$$

Their altitudes, when C is taken as the vertex, are respectively h and $\delta - h$. Hence

$$\frac{h}{\Delta_1} = \frac{\delta - h}{\Delta_2}.$$

On solving this equation for h we find:

3) $$h = \frac{\Delta_1}{\Delta_1 + \Delta_2}\,\delta.$$

If $f(x_1) > \phi(x_1)$ and $f(x_2) < \phi(x_2)$, the result still holds, for Δ_1 and Δ_2 now become negative, but their numerical values

correspond to the lengths of the sides of the triangles in question.

It is easy to express in words the result embodied in 3).

RULE. *In order to see what fraction of $\delta = x_2 - x_1$ must be added to x_1 in order to give X, form the differences*

$$\phi(x_1) - f(x_1), \qquad f(x_2) - \phi(x_2).$$

Then the fraction is the quotient of the first of these differences by their sum.

In practice, an accurately drawn figure on a large scale will often afford a quicker and sufficiently accurate solution.

Example. Returning to Ex. 1 above, we have:

$$f(x) = \cos x, \qquad \phi(x) = x;$$

$$\delta = x_2 - x_1 = .0029, \qquad x_1 = .7389, \qquad x_2 = .7418.$$

$$\phi(x_1) - f(x_1) = -.0004; \qquad f(x_2) - \phi(x_2) = -.0045.$$

$$\frac{.0004}{.0049} .0029 = \frac{.0116}{49} = .0002.$$

Hence the value of the new approximation is

$$X = .7389 + .0002 = .7391.$$

The student will have no difficulty in completing Ex. 2 above in a similar manner. It turns out that the correction is here less than one tenth of δ, and hence it does not influence the second place of decimals: $x' = 1.30$.

Newton's Method. If a higher degree of accuracy is desired, it is well now to apply Newton's method to the function

$$F(x) = f(x) - \phi(x).$$

In the case of Ex. 1 above it is pretty clear that we already have four-place accuracy, and the computation of $F(x)$ for the value $X = .7391$ would only verify the result. This is as far as we can go with four-place tables. If we needed greater

accuracy, we should use Newton's method and five or six-place tables.

Example 2 has been carried only to two-place accuracy, or three significant figures. We can obtain two further figures with the tables at our disposal.

$$y = F(x) = \tan x - e^x.$$

$$y_1 = F(1.30) = 3.602 - 3.669 = -.067.$$

$$\frac{dy}{dx} = \sec^2 x - e^x, \qquad \left.\frac{dy}{dx}\right|_{x=1.30} = 13.97 - 3.67 = 10.30$$

$$X = 1.30 + \frac{.067}{10.3} = 1.3067.$$

To test this result, however, would require five-place tables.

EXERCISES

Solve the following equations:

1. $\cot x = x, \qquad 0 < x < \pi.$ 2. $e^x + \log x = 1.$

3. The hyperbolic sine (sh x or sinh x) and cosine (ch x or cosh x) are defined as follows:

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}, \qquad \operatorname{ch} x = \frac{e^x + e^{-x}}{2},$$

and are tabulated in Peirce's Tables, pp. 120–123. By means of these, reduce the treatment of Ex. 2, § 4, to the methods of the present paragraph.

6. Successive Approximations. We come now to one of the most important of all the methods of numerical computation. In physics it is known as the method of Trial and Error; in mathematics it goes under the name of the method of Successive Approximations.

The problem is that of solving a pair of simultaneous equations,

1) $$F(x, y) = 0, \qquad \Phi(x, y) = 0.$$

The cases which arise in practice are characterized in general by two things: First, there is only one solution of the equations which interests us, and the physical problem enables us to make a fairly good guess at it for the first approximation. Secondly, each of the equations 1) is simple, the curve can readily be plotted in character, and the equation can be solved with ease numerically for the dependent variable when a numerical value has been given to the independent variable. But elimination of one of the unknowns, though sometimes possible, is not expedient, since the resulting equation is hard to solve.

The method is as follows. Plot the curves 1) in character with sufficient accuracy to determine which of them is steeper (*i.e.* has the numerically larger slope) at their point of intersection. Let

C_1: $\qquad F(x, y)=0 \qquad$ or $\qquad y=f(x)$

be the one that is less steep,

C_2: $\qquad \Phi(x, y)=0 \qquad$ or $\qquad x=\phi(y)$,

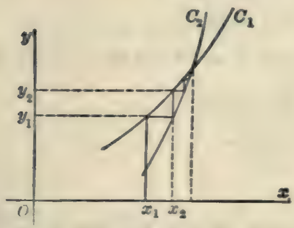


FIG. 56

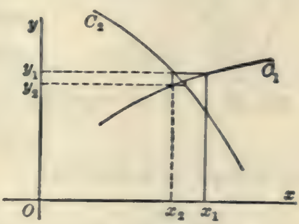


FIG. 57

the other. Then, making the best guess we can to start with, $x=x_1$, compute y_1 from the equation of C_1:

$$y_1=f(x_1),$$

and substitute this value in the equation of C_2, thus getting the second approximation:

$$x_2=\phi(y_1).$$

Proceeding with x_2 in the same manner, we obtain first y_2, then x_3, and so on.

The successive steps of the process are shown geometrically by the broken lines of the figures.

The success of the method depends on the ease with which y can be determined when x is given in the case of C_1, while for C_2 x must be easily attainable from y. If the curves happened to have slopes numerically equal but opposite in sign, the process would converge slowly or not at all. But in this case the arithmetic mean of x_1 and x_2 will obviously give a good approximation.

The method has the advantage that each computation is independent of its predecessor. An error, therefore, while it may delay the computation, will not vitiate the result.

Example. A beam 1 ft. thick is to be inserted in a panel 10×15 ft. as shown in the figure. How long must the beam be made?

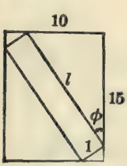


FIG. 58

We have:

$$\begin{cases} \sin\phi + l\cos\phi = 15, \\ \cos\phi + l\sin\phi = 10. \end{cases}$$

Hence $$\cos^2\phi - \sin^2\phi = 10\cos\phi - 15\sin\phi.$$

Now an expression of the form

$$a\cos\phi - b\sin\phi$$

can always be written as

$$\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\phi - \frac{b}{\sqrt{a^2+b^2}}\sin\phi\right) = \sqrt{a^2+b^2}\cos(\phi+\alpha),$$

where $$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

In the present case, then,

$$\cos 2\phi = \sqrt{325}\cos(\phi+\alpha),$$

where $$\cos\alpha = \frac{10}{\sqrt{325}}, \qquad \sin\alpha = \frac{15}{\sqrt{325}}.$$

Thus α is an angle of the first quadrant and

$$\tan\alpha = \tfrac{3}{2}, \qquad \alpha = 56^\circ\ 19'$$

Our problem may be formulated, then, as follows: To find the abscissa of the point of intersection of the curves:

$$y = \cos 2\phi, \qquad y = \sqrt{325}\cos(\phi + \alpha).$$

We know from the figure a good approximation to start with, namely:

$$\tan\phi = \tfrac{2}{3}, \qquad \phi = 33^\circ\ 44'.$$

For this value of ϕ the slopes are given by the equations: *

$$\frac{180}{\pi}\cdot\frac{dy}{d\phi} = -2\sin 2\phi = -2\sin 67^\circ\ 28' = -1.8,$$

$$\frac{180}{\pi}\cdot\frac{dy}{d\phi} = -\sqrt{325}\sin(\phi+\alpha) = -\sqrt{325} = -18.$$

Hence we have:

C_1: $\qquad y = \cos 2\phi;$

C_2: $\quad y = \sqrt{325}\cos(\phi+\alpha) \quad$ or $\quad \phi = \cos^{-1}\dfrac{y}{\sqrt{325}} - \alpha.$

Beginning with the approximation

$$\phi_1 = 33^\circ\ 44',$$

we compute $\qquad y_1 = \cos 67^\circ\ 28' = .3832.$

Passing now to the curve C_2, we compute its ϕ when its $y = y_1$:

$$.3832 = \sqrt{325}\cos(\phi_2+\alpha), \qquad \phi_2 = 32^\circ\ 31'.$$

We now repeat the process, beginning with $\phi_2 = 32^\circ\ 31'$ and find:

$$y_2 = \cos 65^\circ\ 02' = .4221,$$

$$.4221 = \sqrt{325}\cos(\phi_3+\alpha), \qquad \phi_3 = 32^\circ\ 23'.$$

A further repetition gives $\phi_4 = 32^\circ\ 22'$, and this is the value of the root we set out to determine.

* Since the degree is here taken as the unit of angle, the formulas of differentiation involve the factor $\pi/180$; cf. Chap. V, § 2.

EXERCISES

1. Solve the same problem for a beam 2 ft. thick.

2. A cord 1 ft. long has one end fastened at a point O 2 ft. above a rough table, and the other end is tied to a rod 2 ft. long. How far can the rod be displaced from the vertical through O and still remain in equilibrium when released?

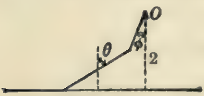


Fig. 59

The equations on which the solution depends are:

$$\begin{cases} 2\cot\theta + \dfrac{1}{\mu} = \cot\phi, \\ 2\cos\theta + \cos\phi = 2. \end{cases}$$

If the coefficient of friction $\mu = \frac{1}{2}$, find the value of ϕ.

3. A heavy ring can slide on a smooth vertical rod. To the ring is fastened a weightless cord of length $2a$, carrying an equal ring knotted at its middle point and having its further end made fast at a distance a from the rod. Find the position of equilibrium of the system.

4. Solve Example 2, § 4, by the method of successive approximations.

7. Arrangement of the Numerical Work in Tabular Form. In the foregoing paragraphs we have laid the chief stress on setting forth the great ideas which underlie these powerful methods of numerical computation. There are, however, certain details of technique which are important, not only for ease in keeping in view the results obtained, but also for accuracy, since they reduce the numerical work to a system. We will illustrate what we mean by an example.

Example. Let it be required to find all the values of x between 0° and 360° which satisfy the equation

$$\sin x = \log_{10}(1 - \cos x).$$

A free-hand graph of each of the functions

1) $y = \sin x$, $y = \log_{10}(1 - \cos x)$

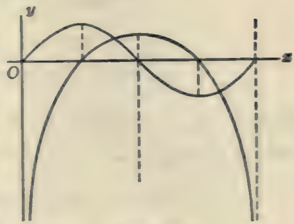


Fig. 60

shows that there is one root between 0° and 180° and a second between 180° and 360°. But these roots cannot be located with any great accuracy in this manner. It is necessary to do exact table work, and to keep the successive results in such form that they are convenient for later reference.

To this end such a table as the following is useful.* Begin with the trial value $x = 150°$.

x	150°				
$\cos x$ $1 - \cos x$ $\log_{10}(1 - \cos x)$	$-.8660$ 1.8660 .2709				
$\sin x$	.5000				

Since the ordinate of the sine curve is larger than that of the logarithmic curve, it is clear from the figure that x is too small. Try $x = 160°$.

Before proceeding further let us ask ourselves whether the above scheme is the simplest for the example in hand. For the special value $x = 150°$ we know $\cos x$ without reference to the tables, and hence one entry of the tables was sufficient. But when $x = 160°$, it will be necessary to enter the tables first for $\cos x$, a second time for $\log_{10}(1 - \cos x)$, and still a third time for $\sin x$.

* Paper ruled in small squares is convenient for these tables, the individual digits being written in separate squares.

Now,
$$1 - \cos x = 2 \sin^2 \frac{x}{2},$$
$$\log_{10} (1 - \cos x) = \log_{10} \sin^2 \frac{x}{2} + \log_{10} 2$$
$$= 2 \log \sin \frac{x}{2} + .3010.$$

Hence it is possible to get along with only two entries of the tables if we make use of the following scheme.

x	160°	164°	163° 3′
$\frac{1}{2} x$	80°	82°	81° 32′
$\log_{10} \sin \frac{1}{2} x$	$\bar{1}.9934$	$\bar{1}.9958$	$\bar{1}.9952$
$2 \log_{10} \sin \frac{1}{2} x$	$\bar{1}.9868$	$\bar{1}.9916$	$\bar{1}.9904$
$+ .3010$	.2878	.2926	.2914
$\sin (180 - x)$	.3420	.2756	.2916

The ordinate of the sine curve is still in excess, but only slightly so. Try $x = 164°$. It is seen that the curves have now crossed. Moreover, the two approximations for x — namely, 160° and 164° — are so near together that we can with advantage apply the method of interpolation of § 5. We have:

$$\phi(x) = \sin x, \qquad f(x) = \log_{10} (1 - \cos x);$$
$$x_1 = 160°, \qquad x_2 = 164°, \qquad \delta = 4°;$$
$$\phi(x_1) = .3420, \qquad f(x_1) = .2878, \qquad \Delta_1 = .0542;$$
$$\phi(x_2) = .2756, \qquad f(x_2) = .2926, \qquad \Delta_2 = .0170;$$
$$h = \frac{\Delta_1}{\Delta_1 + \Delta_2} \delta = \frac{.0542}{.0712} 4 = 3.05.$$

Thus the correction is seen to be 3.05°, or 3° 3′, and the new approximation is:

$$x = 163° \; 3'.$$

For this value of x the values of the two functions, $f(x)$ and $\phi(x)$, differ by a quantity which is comparable with the error of the tables, and the problem is solved.

EXERCISES

1. Determine the other root in the above problem.

2. Solve the equation:

$$\cot x = \log_{10}(1 + \sin x), \qquad 0 < x < 90°.$$

3. Find the positive root of the equation

$$e^{-x} = x^3 - x.$$

Suggestion. Tabulate x, x^3 (from a table of cubes), $x^3 - x$, and e^{-x}.

8. Algebraic Equations. By an *algebraic equation* is meant an equation of the form

$$1) \qquad a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad a_0 \neq 0,$$

where n denotes a positive integer.

If the coefficients a_0, a_1, $\cdots$ are numerical, the roots can be approximated to by the method of interpolation or by Newton's method. In either case it becomes necessary to compute the value of the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

for several values of x, the later ones of which will be at least three- or four-place numbers. There are labor-saving devices for performing these computations, to which we now turn.

Numerical Computation of Polynomials. Let a cubic polynomial, for example, be given:

$$f(x) = ax^3 + bx^2 + cx + d,$$

and let it be required to compute $f(x)$ for the value $x = m$. Write down the following scheme:

$$\begin{array}{cccc} a & am + b & am^2 + bm + c & f(m) \\ \hline am & am^2 + bm & am^3 + bm^2 + cm & \end{array},$$

the explanation of which is as follows. Begin with the first coefficient, a, and multiply it by m to get the expression am which stands below the line. To this expression add the second coefficient, b, to get the second expression above the line, $am+b$. Next, multiply this expression by m to get the expression which stands below it, and continue the process. The last entry above the line will be the required value,

$$f(m) = am^3 + bm^2 + cm + d.$$

Example. Let

$$f(x) = 7x^3 - 6x^2 + 3x - 7,$$

and let it be required to compute the value of $f(x)$ for $x = .8$. Here, the scheme is as follows:

$$\begin{array}{cccc} 7 & -.4 & 2.68 & -4.856 \\ \hline 5.6 & -.32 & 2.144 & \end{array}$$

and hence

$$f(.8) = -4.856.$$

It will be observed that the process requires only additions (or subtractions) and multiplications. The former can be performed mentally. The latter are executed most simply by one of the machines now in general use with computers. These instruments, combined with the method of this paragraph, have rendered Horner's method for solving numerical algebraic equations obsolete.

EXERCISE

Compute the value of

$$5.1x^4 - 3.42x^2 + 1.432x + .8543$$

for $x = .1876$.

In the problems which arise in physics, however, it is not a question of computing all the roots of a numerical equation, about which nothing is known beyond the coefficients. Usually, the equation is a cubic or biquadratic, and only one root is required. Moreover, from the nature of the problem, a close

guess at the value of this root can be made at the outset. Then the methods set forth in this paragraph and in §§ 2, 3 lead quickly to the desired result.

EXERCISES

Solve the following equations, being given that there is one root, and only one, between 0° and 90°:

1. $$4\cos^3\theta - 3\cos\theta = .5283, \qquad 0° < \theta < 90°.$$

2. $$\sin^3\theta - .75\sin\theta = .1278, \qquad 0° < \theta < 90°$$

3. Find the root of the equation

$$x^4 + 2.6x^3 - 5.2x^2 - 10.4x + 5.0 = 0$$

which lies between 0 and 1.

4. Find the root of the equation

$$3x^4 - 12x^3 + 12x^2 - 4 = 0$$

which lies between 2 and 3.

9. Continuation. Cubics and Biquadratics. Aside from the special problem of numerical computation, the simpler algebraic equations present an intrinsic interest which should not be ignored.

Transformations. a) Let the cubic equation

$$f(x) = ax^3 + bx^2 + cx + d = 0, \qquad a \neq 0, \tag{1}$$

be given, and let x be replaced by y, where

$$y = x - h, \qquad x = y + h. \tag{2}$$

Then

$$\begin{aligned} f(x) &= a(y+h)^3 + b(y+h)^2 + c(y+h) + d = \phi(y) \\ &= ay^3 + (3ah + b)y^2 + \cdots, \end{aligned}$$

where the later coefficients are easily written down.

If $y = \beta$ is a root of the equation

3) $$\phi(y) = 0,$$

then $$x = \beta + h$$

will be a root of equation 1). For, it is always true that

$$f(x) = \phi(y)$$

when x and y are connected by the relation 2).

Here, h is any number we please. In particular, h can always be so chosen that the coefficient of the second term of 3) will drop out. It is sufficient to set

4) $$3ah + b = 0, \qquad \text{or} \qquad h = -\frac{b}{3a}.$$

Obviously, the same method can be used to transform an algebraic equation of any degree into a new equation whose second term is lacking.

EXERCISES

Transform the following equations into equations in which the second term is lacking.

1. $x^3 + x^2 - x + 1 = 0.$

2. $3x^3 - 4x^2 + 2 = 0.$

3. $x^4 + x^3 - x^2 + 1 = 0.$

4. $5x^4 - 4x^3 + x^2 + x - 80 = 0.$

5. $3x^4 - 7x^3 + x^2 - x - 1 = 0.$

6. $x^6 + x^5 + x^2 + x + 1 = 0.$

b) Let the equation

5) $$f(x) = x^4 + px^2 + qx + r = 0$$

be given, and let x be replaced by y, where

6) $$y = \frac{x}{k}, \qquad x = ky.$$

Then

$$f(x) = k^4y^4 + k^2py^2 + kqy + r.$$

Denoting this last polynomial by $\phi(y)$, we have

$$f(x) = \phi(y)$$

for all values of x and y which are connected by the relation 6)

It is clear that, if $y = \beta$ is a root of the equation

7) $$\phi(y) = 0,$$

then $x = k\beta$ will be a root of 5).

The factor k is arbitrary, and we can always determine it so that, on dividing equation 7) through by k:

$$y^4 + \frac{p}{k^2} y^2 + \frac{q}{k^3} y + \frac{r}{k^4} = 0,$$

the coefficient of y^2 will be numerically equal to unity (provided that $p \neq 0$):

i) $$\frac{p}{k^2} = 1 \quad \text{or} \quad k = \sqrt{p}, \quad \text{if} \quad p > 0;$$

ii) $$\frac{p}{k^2} = -1 \quad \text{or} \quad k = \sqrt{-p}, \quad \text{if} \quad p < 0.$$

In this way, equation 5) can be reduced to one of the two forms

α) $$y^4 + y^2 + Ay + B = 0;$$

β) $$y^4 - y^2 + Ay + B = 0.$$

If, in particular, $p = 0$ and $q \neq 0$, 5) can be reduced to the form

γ) $$y^4 + y + B = 0.$$

The method can be applied to any algebraic equation whose second term is lacking:

$$x^n + c_2 x^{n-2} + c_3 x^{n-3} + \cdots + c_n = 0.$$

EXERCISES

1. Replace the equation

$$7x^4 - 175x^2 + 16x + 10 = 0$$

by an equation of the type β), and state precisely the relation of the roots of the second equation to those of the first.

2. Show that, if in the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

where $a_0 \neq 0$ and $a_n \neq 0$, the transformation

$$y = \frac{1}{x}$$

is made, the roots of the new equation,

$$a_ny^n + a_{n-1}y^{n-1} + \cdots + a_0 = 0$$

are the reciprocals of the roots of the given equation.

3. If on transforming equation 1) by 2), where h is determined by 4), the constant term in the resulting equation 3), $\phi(y) = 0$, does not vanish, the further transformation

$$8) \qquad y = \frac{1}{z}, \qquad \text{or} \qquad x = \frac{1}{z} + h,$$

will carry 1) into an equation in which the linear term is lacking:

$$Az^3 + Bz^2 + D = 0. \qquad A \neq 0, \quad D \neq 0.$$

The theorem holds in full generality for an algebraic equation of any higher degree. State it accurately.

4. Replace the equation

$$x^4 - 4x^3 - 6x^2 + 16x - 4 = 0$$

by an equation of the type

$$Ay^4 + By^3 + Cy^2 + D = 0.$$

Graphical Treatment. We have already seen that the cubic

$$x^3 + px + q = 0$$

can be solved graphically by cutting the standard graph

$$y = x^3$$

by the straight line,

$$y = -px - q.$$

Since the general cubic can be reduced by the transformation 2) to a cubic of this type, we may consider the general problem of the graphical solution of a cubic as solved.

To obtain a similar solution for the general biquadratic,

9) $$ax^4 + bx^3 + cx^2 + dx + e = 0, \qquad a \neq 0,$$

begin by reducing it to one of the three forms:

i) $$y^4 + y^2 + Ay + B = 0;$$

ii) $$y^4 - y^2 + Ay + B = 0;$$

iii) $$y^4 \qquad + Ay + B = 0.$$

An equation of type i):

$$x^4 + x^2 + Ax + B = 0,$$

can be solved graphically by cutting the standard curve

$$y = x^4$$

by the parabola

$$y = -x^2 - Ax - B.$$

A similar procedure leads to a solution in the case of each of the other two types, ii) and iii).

The Method of Curve Plotting. Let the coefficients a, e in equation 9) be different from 0. By means of Ex. 3, p. 192, the equation can be reduced to one of the following type:

$$Ax^4 + Bx^3 + Cx^2 + E = 0.$$

In order to discuss the number and location of the roots of this equation, it is sufficient to plot the curve

$$y = Ax^4 + Bx^3 + Cx^2 + E.$$

Since all the maxima, minima, and points of inflection of this curve can be determined by means, at most, of quadratic equations, the problem is readily solved in any given numerical case.

EXERCISES

Determine the number of real roots of each of the following equations, and locate them approximately.

1. $3x^4 + 8x^3 - 90x^2 + 100 = 0.$

2. $3x^4 + 8x^3 - 90x^2 + 500 = 0.$

3. $3x^4 + 8x^3 - 90x^2 + 1500 = 0.$

4. Show that the equation

$$3x^4 + 4x^3 + 2x^2 + 1 = 0$$

has no real roots.

How many real roots has each of the following equations?

5. $x^5 - 5x - 1 = 0.$

6. $x^3 + 7x - 1 = 0.$

7. $x^3 - 4x + 1 = 0.$

8. $x^3 - 3x - 2 = 0.$

9. $x^3 - x + 3 = 0.$

10. $4x^3 - 15x^2 + 12x + 1 = 0.$

11. $3x^4 + 4x^3 + 6x^2 - 1 = 0.$

12. $3x^4 - 4x^3 + 12x^2 + 7 = 0.$

13. How many positive roots has the equation

$$6x^4 + 8x^3 - 12x^2 - 24x - 1 = 0?$$

14. Has the equation

$$3x^8 - 8x^6 + 12x^3 + 1 = 0$$

any real roots?

15. By means of the graph of the function

$$y = x^3 + px + q$$

show that the equation

$$x^3 + px + q = 0$$

has

(*a*) 1 real root when $\dfrac{p^3}{27} + \dfrac{q^2}{4} > 0;$

(*b*) 3 real roots when $\dfrac{p^3}{27} + \dfrac{q^2}{4} < 0;$

(*c*) 2 real roots when $\dfrac{p^3}{27} + \dfrac{q^2}{4} = 0,$ $\{p \text{ and } q \text{ not both } 0\}$

(*d*) 1 real root when $\dfrac{p^3}{27} + \dfrac{q^2}{4} = 0,$ $\{p = q = 0\}$

In case (*c*) it is customary to count one of the roots twice; in case (*d*), to count the root three times.

16. Extend the criterion of Ex. 15 to the case of the general cubic

$$ax^3 + bx^2 + cx + d = 0.$$

10. Curve Plotting. We will close this chapter by considering the application of the principles set forth in the earlier paragraph on curve plotting (Chap. III, § 5) to some interesting curves of a more complex nature.

Example 1. To plot the curve

1) $$y = \frac{1}{x-1} + \frac{1}{x+1}.$$

The curve is obviously not symmetric in either axis; but the test for symmetry in the origin is fulfilled, since on replacing x by $-x$ and y by $-y$ the new equation,

$$-y = \frac{1}{-x-1} + \frac{1}{-x+1}$$

is equivalent to the original equation, 1). Incidentally we observe that the curve passes through the origin.

In consequence of the symmetry just noted it will be sufficient to plot the curve for positive values of x and then rotate the figure about the origin through 180°.

To each positive value of x but one there corresponds one value of y. When x approaches 1 as its limit from above (*i.e.* always remaining greater than 1), y becomes positively infinite. Hence the line $x = 1$ is an asymptote for one branch of the curve.

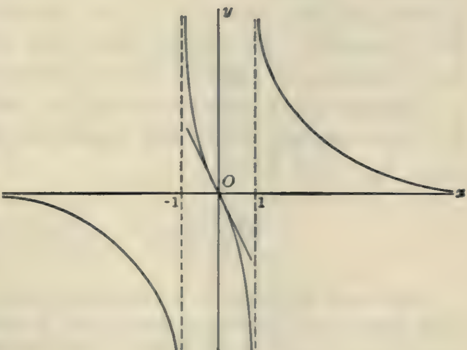


FIG. 61

When x approaches 1 from below, y becomes negatively infinite, and hence this same line, $x=1$, is an asymptote for a second branch of the curve.

For all other positive values of x, y is continuous.

The slope of the curve is given by the equation

$$2)\qquad \frac{dy}{dx}=-\left(\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}\right),$$

and is seen to be negative for all values of x for which y is continuous. Thus, in particular, the curve is seen to have no maxima or minima, or in fact any points at which the tangent is horizontal.

The second derivative is given by the formula

$$3)\qquad \frac{d^2y}{dx^2}=2\left(\frac{1}{(x-1)^3}+\frac{1}{(x+1)^3}\right).$$

When $x>1$, the right-hand side of this equation is always positive, and so the curve is concave upward in this interval Moreover, it is evident from 1) that, when $x=+\infty$, y approaches 0 from above, and so the positive axis of x is also an asymptote.

In the interval $0<x<1$, the second derivative is surely sometimes negative, for this is obviously the case when x is only slightly less than 1. Is d^2y/dx^2 always negative in this interval? If not, it must pass through the value 0; for a continuous function cannot change from a positive to a negative value without taking on the intermediate value 0.* Let us set, then, the right-hand side of equation 3) equal to 0 and solve:

$$2\left(\frac{1}{(x-1)^3}+\frac{1}{(x+1)^3}\right)=0.$$

* How must the graph of a continuous function look, which is sometimes positive and sometimes negative? It must cross the axis of abscissas, must it not? At the point or points where it crosses, the function has the value 0.

This equation is equivalent to the following:

$$\frac{1}{(x-1)^3} = -\frac{1}{(x+1)^3}.$$

Extracting the cube root of each side of this equation, we have:

$$\frac{1}{x-1} = -\frac{1}{x+1}.$$

Clearing of fractions we find:

$$x+1 = -(x-1),$$

or

$$2x = 0.$$

Hence $x = 0$ is the only value of x for which d^2y/dx^2 can vanish, and we see at once that the right-hand side of 3) does vanish for $x = 0$.

We have thus proven that the continuous function 3) is nowhere 0 in the interval $0 < x < 1$, and since it is negative in part of this interval, it is negative throughout. Hence the curve is concave downward throughout the interval.

It is now easy to complete the graph. The curve has one point of inflection, — namely, the origin, — and the slope there is, by 2), equal to -2.

EXERCISES

Plot the following curves:

1. $y = \frac{3}{3+x^2}$.

2. $y = \frac{3x}{3+x^2}$.

3. $y = \frac{1}{x-2} + \frac{1}{x+2}$.

4. $y = \frac{1}{x} + \frac{1}{x-1}$.

5. $y = \frac{1}{x} + \frac{1}{x+1}$.

6. $y = \frac{1}{x^2-1}$.

7. $y = \frac{1}{x^2}$.

8. $y = \frac{1}{(1-x)^2}$.

9. $y = \frac{1}{x^3}$.

10. $y = \frac{1}{(x+1)^3}$.

11. $y = x + \frac{1}{x}$. **12.** $y = x - \frac{1}{x}$.

13. $y = \frac{4}{1-x} + x^2 - 2x$. **14.** $y = \frac{3}{3+x} - 6x - x^2$.

15. $y = \frac{1}{x-1} - \frac{1}{x+1}$. **16.** $y = \frac{1}{x} - \frac{1}{x-1}$.

Example 2. To plot the curve

4) $$y^2 = x^2 + x^3.$$

We observe first of all that the curve is symmetric in the axis of x. It is sufficient, therefore, to plot the curve for positive values of y, and then fold this part of the curve over on the axis of x. The curve goes through the origin.

Unlike the examples hitherto considered, this curve does not permit an arbitrary choice of x. It is only when the right-hand side is positive or zero, *i.e.* when

$$x^2 + x^3 \geqq 0,$$

or

$$x^2(1+x) \geqq 0 \qquad \text{or} \qquad x \geqq -1,$$

that there will be a corresponding value of y and thus a point with the given abscissa.

Obviously, the curve cuts the axis of x at the origin and at the point $x = -1$. We have, then, essentially two problems:

i) to plot the curve for $x > 0$;
ii) to plot the curve for $-1 < x < 0$.

i) When $x > 0$, the positive value of y is given by the equation

5) $$y = x\sqrt{1+x}.$$

Hence

6) $$\frac{dy}{dx} = \frac{2+3x}{2\sqrt{1+x}}.$$

For positive values of x the right-hand side of this equation is always positive, and hence there are no horizontal tangents in the interval under consideration; the slope of this part of the curve is always positive. In particular, the slope at the origin is unity:

$$\left.\frac{dy}{dx}\right|_{x=0}=1.$$

The second derivative has the value

$$7)\qquad \frac{d^2y}{dx^2}=\frac{4+3x}{4(1+x)^{\frac{3}{2}}}.$$

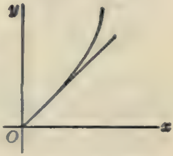


FIG. 62

The right-hand side of this equation is always positive in this interval, and thus it appears that the curve is concave upward for all positive values of x.

ii) When $-1<x<0$, the positive value of y is no longer given by the formula 5), since x is now negative.* In the present case,

$$8)\qquad y=-x\sqrt{1+x},$$

and consequently

$$9)\qquad \frac{dy}{dx}=-\frac{2+3x}{2\sqrt{1+x}},$$

$$10)\qquad \frac{d^2y}{dx^2}=-\frac{4+3x}{4(1+x)^{\frac{3}{2}}}.$$

The first derivative will vanish if, and only if,

$$2+3x=0,$$

or

$$x=-\tfrac{2}{3}.$$

* The student must have clearly in mind the definition of the function expressed by the $\sqrt{}$ sign, which was laid down in Chap. I, § 1. This function is the *positive* square root of the radicand; it can never take on a negative value.

It is, therefore, important to determine the corresponding point on the curve and draw the tangent there:

$$y\big|_{x=-\frac{2}{3}}=-(-\tfrac{2}{3})\sqrt{1-\tfrac{2}{3}}=\frac{2\sqrt{3}}{9}=.38.$$

Two other important points for the present curve are the origin and the point $x=-1$, $y=0$. At these points the slope has the following values:

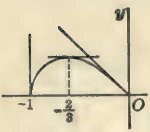


FIG. 63

$$\left.\frac{dy}{dx}\right|_{x=0}=-1; \qquad \left.\frac{dy}{dx}\right|_{x=-1}=\infty.$$

Draw the corresponding tangents.

From the expression 10) for the second derivative it is clear that, when $-1<x<0$, the right-hand side of this equation is always negative, and hence the curve is concave downward throughout the whole interval in question. We can now draw in the curve in this interval, Fig. 63.

The curve is now complete above the axis of x. It remains, therefore, merely to fold this part over on that axis. The entire curve is shown in Fig. 64.

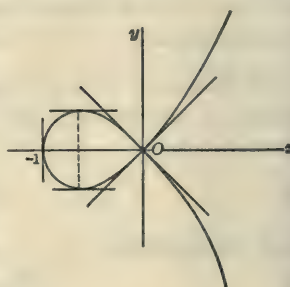


FIG. 64

EXERCISES

Plot the following curves:

1. $y^2=x^2-x^3$.

2. $y^2=x-2x^2+x^3$.

3. $$y^2=(x-a)^2(Ax+B)$$

Suggestion: Write the second factor in the form

$$Ax+B=A(x-b), \text{ where } b=\frac{B}{A},$$

and make two cases: i) $A>0$; ii) $A<0$. Discuss the omitted case, $A=0$.

4. $y^2 = x^2 - x^4$. 5. $y^2 = x^2 + x^4$.

Example 3. To plot the curve

$$y^2 = x(x-1)(x-2). \tag{11}$$

The curve lies wholly in the regions

$$0 \leqq x \leqq 1 \qquad \text{and} \qquad 2 \leqq x.$$

It is symmetric in the axis of x, and hence it is sufficient to plot it for positive values of y.

The function

$$y = \sqrt{x(x-1)(x-2)}$$

is continuous in the interval $0 \leqq x \leqq 1$. It starts with the value 0 when $x = 0$, increases, and finally decreases to 0 when $x = 1$.

When x, starting with the value 2, increases, y, starting with the value 0, increases, always remaining positive, and increasing without limit as x becomes infinite.

Fig. 65

So much from considerations of continuity. A more specific discussion of the character of the curve can be given by means of the derivatives of the function.

The slope is given by the formula

$$2y\frac{dy}{dx} = 3x^2 - 6x + 2 \tag{12}$$

or

$$\frac{dy}{dx} = \frac{3x^2 - 6x + 2}{2\sqrt{x(x-1)(x-2)}}. \tag{13}$$

The slope is infinite when $x = 0$ or 1:

$$\left.\frac{dy}{dx}\right|_{x=0} = \infty, \qquad \left.\frac{dy}{dx}\right|_{x=1} = \infty.$$

At these points, the tangent is vertical.

The slope is 0 when

$$3x^2 - 6x + 2 = 0.$$

The roots of this equation are

$$x = 1 + \frac{1}{\sqrt{3}}, \qquad x = 1 - \frac{1}{\sqrt{3}}.$$

The first of these values does not correspond to any point on the curve. The second, $x = .42$, yields a horizontal tangent, the ordinate being

$$y = \sqrt{\frac{2}{3\sqrt{3}}} = .62.$$

Plot this point and draw the tangent. From the above discussion on the basis of continuity it is obvious that this point must be a maximum, and we see that there are no other maxima or minima. But it is not clear that the curve has no points of inflection in this interval.

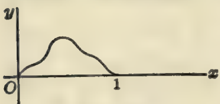


Fig. 66

To treat this question, compute the second derivative. This might be done by means of formula 13); but it is simpler to use 12):

$$2y\frac{d^2y}{dx^2} + 2\frac{dy^2}{dx^2} = 6x - 6,$$

$$y\frac{d^2y}{dx^2} = 3x - 3 - \frac{dy^2}{dx^2}.$$

Substitute here the value of dy/dx from 13) and reduce:

14) $$y\frac{d^2y}{dx^2} = \frac{3x^4 - 12x^3 + 12x^2 - 4}{4x(x-1)(x-2)}.$$

And now we seem to be in difficulty. How are we going to tell when d^2y/dx^2 is positive, when negative?

First of all, y is positive, and so the sign of d^2y/dx^2 will be the same as that of the right-hand side of the equation.

Secondly, in the interval in question, $0 < x < 1$, the denominator is positive.

All turns, then, on whether the numerator, *i.e.* the function

15) $$u = 3x^4 - 12x^3 + 12x^2 - 4,$$

is positive or negative. To answer this question, plot the graph of the function 15). The slope of the graph is given by the equation

16) $$\frac{du}{dx} = 12x^3 - 36x^2 + 24x = 12x(x-1)(x-2).$$

In the interval in question, the right-hand side of this last equation is always positive. Hence u increases with x throughout the interval $0 \leqq x \leqq 1$, and consequently attains its greatest value at the end-point, $x = 1$. Here,

$$u\big|_{x=1} = -1.$$

We see, therefore, that u is negative throughout the whole interval in question, and consequently the graph of 1) is concave downward in this interval.

FIG. 67

The reasoning by which we determined whether u is positive or negative is an excellent illustration of the practical application of the methods of curve plotting which we have learned. It is in no wise a question of the precise values of u which correspond to x. The question is merely: Is u positive, or is it negative? Without the labor of a single computation involving table work we have answered this question with the greatest ease. Such questions as these arise again and again in physics, and the aid which the calculus is able to render here is most important.

One further point. It may seem to have been a fluke that we were able to factor the polynomial in 16) and thus simplify so materially the further discussion. And yet, in the problems which arise in practice, — the problems with a *pedigree*, — just such simplifications as this present themselves with great frequency.

To complete the graph, it remains to consider the interval $2 \leqq x < \infty$. Since

$$\left.\frac{dy}{dx}\right|_{x=2} = \infty,$$

the tangent to the curve is vertical at the point where the curve meets the axis of x. It is clear, then, that the curve must be concave downward for a while, and so $d^2y/dx^2 < 0$ for values of x slightly greater than 2. This is verified from 14), since

$$17) \qquad u|_{x=2} = -4.$$

On the other hand, when x is large, u is positive and d^2y/dx^2 is positive. Hence the curve is concave upward. There must be, therefore, a point of inflection in the interval, and there may be several.

From 14) we see that the second derivative will vanish when and only when

$$3x^4 - 12x^3 + 12x^2 - 4 = 0.$$

The problem is, then, to determine the number of roots of this equation which are greater than 2, and to compute them.

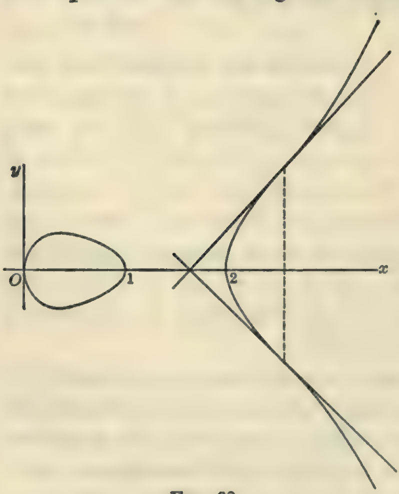


FIG. 68

Again, it is a question of the graph of 15). When $x > 2$, we see from 16) that

$$\left.\frac{du}{dx}\right|_{x>2} > 0.$$

Hence u steadily increases with x. Now, from 17), u starts with a negative value, and u is positive and large when x is large. Hence u vanishes for just one value of x which is greater than 2. Since $u|_{x=3} = 23$, this root is seen to lie between 2 and 3. It can be determined to any required degree of accuracy by the foregoing methods of

this chapter, which find herewith a practical application. To two places of decimals it is 2.47.

EXERCISES

Plot the following curves:

1. $y = x^3 - x.$

2. $y = x - x^3.$

3. $y^2 = x^3 + x.$

4. $y^2 = 1 - x^4.$

5. $y^2 = (x^2 - 1)(x^2 - 4).$

6. $y^2 = (1 - x^2)(x^2 - 4).$

7. $y^2 = \dfrac{1}{x^3 - x}.$

8. $y^2 = \dfrac{1}{(x^2 - 1)(x^2 - 4)}.$

9. $y^2 = \dfrac{x}{1 - x}.$

10. $y^2 = \dfrac{x}{1 + x}.$

11. $y^2 = \dfrac{x^2}{1 + x^2}.$

12. $y^2 = \dfrac{x^2}{1 - x^2}.$

13. $y^2 = \dfrac{x^2}{x - 1}.$

14. $y^2 = \dfrac{x^2}{1 + x}.$

15. $y^2 = x^3 - 4x^2 + 3x.$

16. $y = \sin x + \sin 2x.$

17. $y = \sin x - \sin 2x.$

18. $y = \cos x + \cos 2x.$

19. $y = \cos x - \cos 2x.$

20. $y = x + \sin x, 0 \leqq x \leqq \pi$

CHAPTER VIII

THE INVERSE TRIGONOMETRIC FUNCTIONS

1. Inverse Functions. Let

(1) $$y = f(x)$$

be a given function of x, and let us solve this equation for x as a function of y:

(2) $$x = \phi(y).$$

Then $\phi(y)$ is called the *inverse function*, or the *inverse* of the function $f(x)$. Thus if $f(x) = x^3$, we have

$$y = x^3.$$

Hence $$x = \sqrt[3]{y},$$

and $\phi(y)$ is here the function $\sqrt[3]{y}$.

When the given function is tabulated, the table also serves as a tabulation of the inverse function. It is necessary merely to enter it from the opposite direction. Thus, if we have a table of cubes, we can use it to find cube roots by simply reversing the rôles of the two columns.

In the same way, the graph of the function (1) serves as the graph of the function (2), provided in the latter case we take y as the *independent variable*, and x as the *dependent variable*, or *function*.

The graph of the inverse function, plotted with x as the independent variable, can be obtained from the former graph as follows. Make the transformation of the plane which is defined by the equations:

(3) $$\left.\begin{aligned} x' &= y, \\ y' &= x, \end{aligned}\right\} \qquad \text{or} \qquad \left.\begin{aligned} x &= y', \\ y &= x'. \end{aligned}\right\}$$

It is easy to interpret this transformation. Any point, whose coordinates are (x, y), is carried over into a point (x', y') situated as follows: Draw a line L through the origin bisecting the angle between the positive axes of coordinates. Drop a perpendicular from (x, y) on L and produce it to an equal distance on the other side of L. The point thus determined is the point (x', y'). The proof of this statement is immediately evident from the figure.

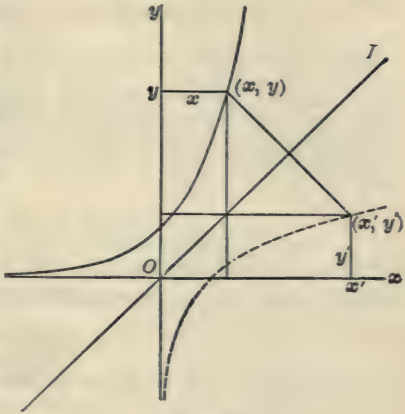


FIG. 69

Thus it appears that the transformation (3) can be generated by rotating the plane about L through 180°.

The transformation is also spoken of as a *reflection in L*, since if a plane mirror were set at right angles to the plane of (x, y) and so that the line L would lie in the surface of the mirror, the image of any figure, as seen in the mirror, would be the transformed figure.

Monotonic Functions. A function, $f(x)$, is said to be *monotonic* if it is single-valued and if, as x increases, $f(x)$ always increases, or else always decreases. We shall be concerned only with functions which are, in general, continuous. It is obvious that the inverse of a monotonic function is also monotonic.

A given function,

$$y = f(x),$$

can in general be considered as made up of a number of pieces, each of which is monotonic in a certain interval.* Thus the function

(4) $$y = x^2$$

* There are functions which do not have this property; but they do not play an important rôle in the elements of the Calculus.

can be taken as made up of two pieces, corresponding respectively to those portions of the graph which lie in the first and the second quadrants, the corresponding intervals for x being here

$$-\infty < x \leqq 0, \qquad 0 \leqq x < \infty.$$

Each of the pieces, of which $f(x)$ is made up, has a monotonic inverse, and thus the function $\phi(x)$ inverse to $f(x)$ is represented by a number of monotonic functions.

In the example just cited, the inverse function is multiple-valued:

$$(5) \qquad y = \pm\sqrt{x}.$$

But one of the two pieces into which the original function was divided yields the single-valued function

$$(6) \qquad y = \sqrt{x},$$

the so-called *principal value* of the multiple-valued function (5); the other,

$$y = -\sqrt{x},$$

the remainder of (5).

The derivative of a monotonic function cannot change sign; but it can vanish or become infinite at special points. Thus

$$y = \sqrt{a^2 - x^2}, \qquad 0 \leqq x \leqq a,$$

is a decreasing monotonic function. Its derivative is, in general, negative; but when $x = 0$, it vanishes, and when $x = a$, it becomes infinite.

Differentiation of an Inverse Function. The function $\phi(x)$ inverse to a given function $f(x)$ can be differentiated as follows. By definition, the two equations

$$(7) \qquad y = \phi(x) \qquad \text{and} \qquad x = f(y)$$

are equivalent; they are two forms of one and the same relation between the variables x and y. Their graphs are identical.

Take the differential of each side of the second equation:

$$dx = df(y) = D_y f(y) \cdot dy.$$

Hence

(8) $$\frac{dy}{dx} = \frac{1}{D_y f(y)}.$$

To complete the formula, express the right-hand side of (8) in terms of x by means of (7).

2. The Inverse Trigonometric Functions. The inverse trigonometric functions are chiefly important because of their application in the Integral Calculus. They are defined as follows.

(*a*) *The Function* $\sin^{-1} x$. The inverse of the function

(1) $$y = \sin x$$

is obtained as explained in § 1 by solving this equation for x as a function of y, and is written:

(1′) $$x = \sin^{-1} y,$$

read "the anti-sine of y." * In order to obtain the graph of the function

(2) $$y = \sin^{-1} x$$

we have, then, merely to reflect the graph of (1) in the bisector of the angle made by the positive coordinate axes. We are thus led to a multiple-valued function, since the line $x = x'(-1 \leqq x' \leqq 1)$ cuts the graph in more than one point, — in fact, in an infinite number of points. For most purposes of the Calculus, however, it is allowable and advisable to pick

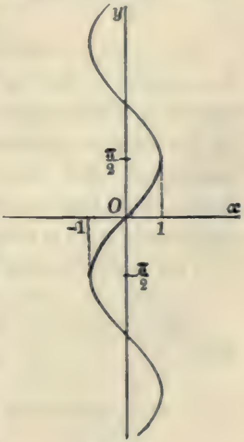


FIG. 70

* The usual notation on the Continent for $\sin^{-1} x$, $\tan^{-1} x$, etc., is arc sin x, arc tan x, etc. It is clumsy, and is followed for a purely academic reason; namely, that $\sin^{-1} x$ might be misunderstood as meaning the minus first power of $\sin x$. It is seldom that one has occasion to write the reciprocal of $\sin x$ in terms of a negative exponent. When one wishes to do so, all ambiguity can be avoided by writing $(\sin x)^{-1}$.

out just *one* value of the function (2), most simply the value that lies between $y=-\pi/2$ and $y=+\pi/2$, and to understand by $\sin^{-1}x$ the *single-valued* function thus obtained. This determination is called the *principal value* of the multiple-valued function $\sin^{-1}x$. Its graph is the portion of the curve in Fig. 70 that is marked by a heavy line. This shall be our convention, then, in the future unless the contrary is explicitly stated, and thus

$$(3) \qquad y=\sin^{-1}x$$

is equivalent to the relations:

$$(3') \qquad x=\sin y, \qquad -\frac{\pi}{2}\leqq y\leqq\frac{\pi}{2}.$$

In particular,

$$\sin^{-1}0=0, \qquad \sin^{-1}1=\frac{\pi}{2}, \qquad \sin^{-1}(-1)=-\frac{\pi}{2}.$$

The student should now prepare a second plate, showing the graphs of the three functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$. Place the first in the upper left-hand corner of the sheet; the second, in the upper right-hand corner; and the third on the lower half-sheet. All of these curves can be ruled from the templets. Use a fine lead-pencil; then mark in the principal value of the function in a clean, firm red line. Also mark, in each figure, *all* the principal points, as is done in Fig. 70 of the text.

Differentiation of $\sin^{-1}x$. In order to differentiate the function

$$y=\sin^{-1}x,$$

make the equivalent equation,

$$x=\sin y$$

the point of departure. Then

$$dx=d\sin y=\cos y\,dy.$$

Hence

$$\frac{dy}{dx}=\frac{1}{\cos y}.$$

The right-hand side of this equation can be expressed in terms of x as follows. Since

$$\sin^2 y + \cos^2 y = 1$$

and since $\sin y = x$, we have,

$$\cos^2 y = 1 - x^2, \qquad \cos y = \pm\sqrt{1 - x^2}.$$

We have agreed, however, to understand by $\sin^{-1} x$ the principal value of this function. Hence y is subject to the restriction: $-\pi/2 \leqq y \leqq \pi/2$, and consequently $\cos y$ is positive (or zero). We must, therefore, take the upper sign before the radical,* the final result thus being:

(4) $$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}},$$

or

(4′) $$d \sin^{-1} x = \frac{dx}{\sqrt{1 - x^2}}.$$

(*b*) *The Function* $\cos^{-1} x$. The treatment here is precisely similar. The definition is as follows:

(5) $$y = \cos^{-1} x \qquad \text{if} \qquad x = \cos y,$$

read: "anti-cosine x").

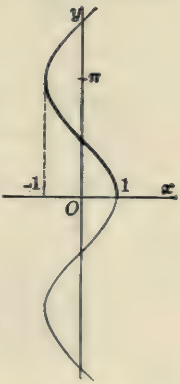


FIG. 71

The graph of the function $\cos^{-1} x$ is as shown in Fig. 71. Like $\sin^{-1} x$, this function is also infinitely multiple-valued. A single-valued branch is selected by imposing the further condition

$$0 \leqq y \leqq \pi.$$

This determination is known as the *principal value* of $\cos^{-1} x$:

(6) $$y = \cos^{-1} x, \qquad 0 \leqq y \leqq \pi.$$

* Geometrically the slope of the portion of the graph in question is always positive, and so we must use the positive square root of $1 - x^2$.

It will be understood henceforth that the principal value is meant unless the contrary is explicitly stated.

In preparing the graph of this function, mark the principal value as a firm red line.

To differentiate the function $\cos^{-1} x$, use the implicit form of equation (5):

$$x = \cos y.$$

Hence

$$dx = d \cos y = -\sin y \, dy$$

and

$$\frac{dy}{dx} = -\frac{1}{\sin y}.$$

For the principal value, $\sin y$ is positive, and hence

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}, \tag{7}$$

or

$$d \cos^{-1} x = -\frac{dx}{\sqrt{1-x^2}}. \tag{7'}$$

The principal values of the functions $\sin^{-1} x$ and $\cos^{-1} x$ are connected by the identical relation:

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}. \tag{8}$$

By means of this relation, the differentiation of $\cos^{-1} x$ could have been performed immediately.

(c) *The Function* $\tan^{-1} x$. Here, the definition is as follows:

$$y = \tan^{-1} x \qquad \text{if} \qquad x = \tan y, \tag{9}$$

(read: "anti-tangent x").

The principal value is defined as that determination which lies between $-\pi/2$ and $\pi/2$:

$$y = \tan^{-1} x, \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}. \tag{10}$$

In preparing the graph of this function, mark the principal value as a firm red line.

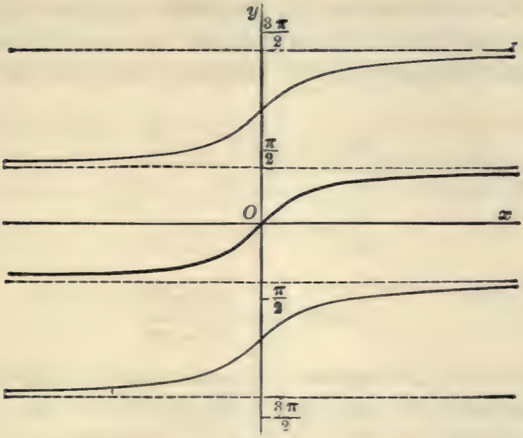


FIG. 72

To differentiate $\tan^{-1} x$ use the implicit form (9). Hence

$$dx = d \tan y = \sec^2 y \, dy,$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

Since $$\sec^2 y = 1 + \tan^2 y$$

and $\tan y = x$, it follows that

(11) $$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2},$$

or

(12) $$d \tan^{-1} x = \frac{dx}{1 + x^2}.$$

(*d*) *The Function* $\cot^{-1} x$. Here, the definition is:

(13) $$y = \cot^{-1} x \qquad \text{if} \qquad x = \cot y,$$

(read: "anti-cotangent x").

The principal value is chosen as that one which lies between 0 and π:

(14) $$y = \cot^{-1} x, \qquad 0 < y < \pi.$$

The differentiation can be performed as in the case of the function $\tan^{-1} x$, but still more simply by means of the identical relation connecting the principal values of $\tan^{-1} x$ and $\cot^{-1} x$:

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}. \tag{15}$$

Hence

$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}, \tag{16}$$

or

$$d\cot^{-1} x = -\frac{dx}{1+x^2}. \tag{17}$$

It is well for the student to make a graph of this function, also, drawing in the principal value, as usual, in red.

The following identity holds for positive values of x, when the principal values of the functions are used:

$$\tan^{-1}\frac{1}{x} = \cot^{-1} x, \qquad 0 < x. \tag{18}$$

For negative values of x it reads:

$$\tan^{-1}\frac{1}{x} = \cot^{-1} x - \pi, \qquad x < 0. \tag{18'}$$

Remarks. The other inverse trigonometric functions, $\sec^{-1} x$, $\csc^{-1} x$, can be treated in a similar manner. They are, however, without importance in practice. Their principal values cannot be defined by means of a single continuous curve. The graph necessarily consists of more than one piece; it is most natural to take it as consisting of two pieces.

Corresponding to the Addition Theorem for each of the trigonometric functions, there are functional relations for the inverse trigonometric functions. Thus, for $\tan^{-1} x$:

$$\tan^{-1} u + \tan^{-1} v = \tan^{-1}\frac{u+v}{1-uv}. \tag{19}$$

These relations, however, are not always true when the principal value of each of the functions is taken, and for this

reason it is usually better not to employ them. If, however, in a particular case, u and v are each numerically less than unity, the principal values can be used throughout in (19).

3. Shop Work. The student will now add to his list of Special Formulas the four new formulas of this chapter. The list of formulas of differentiation is now complete. It reads as follows.

SPECIAL FORMULAS OF DIFFERENTIATION

1. $$dc = 0.$$

2. $$dx^n = nx^{n-1}\,dx.$$

3. $$d\sin x = \cos x\,dx.$$

4. $$d\cos x = -\sin x\,dx.$$

5. $$d\tan x = \sec^2 x\,dx.$$

6. $$d\cot x = -\csc^2 x\,dx.$$

7. $$d\log x = \frac{dx}{x}.$$

8. $$de^x = e^x\,dx.$$

9. $$da^x = a^x \log a\,dx.$$

10. $$d\sin^{-1} x = \frac{dx}{\sqrt{1-x^2}}.$$

11. $$d\cos^{-1} x = -\frac{dx}{\sqrt{1-x^2}}.$$

12. $$d\tan^{-1} x = \frac{dx}{1+x^2}.$$

13. $$d\cot^{-1} x = -\frac{dx}{1+x^2}.$$

It is important that the student gain facility in the use of the new results.

Example 1. Differentiate the function

$$u = \cos^{-1}\frac{x}{a}, \qquad a > 0.$$

Let

$$y = \frac{x}{a}.$$

Then

$$u = \cos^{-1} y,$$

$$du = d\cos^{-1} y$$

$$= -\frac{dy}{\sqrt{1-y^2}};$$

$$dy = \frac{dx}{a}.$$

Hence

$$-\frac{dy}{\sqrt{1-y^2}} = -\frac{\frac{dx}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = -\frac{dx}{\sqrt{a^2-x^2}}$$

and, finally,

$$d\cos^{-1}\frac{x}{a} = -\frac{dx}{\sqrt{a^2-x^2}}.$$

In abbreviated form,

$$d\cos^{-1}\frac{x}{a} = -\frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = -\frac{dx}{\sqrt{a^2-x^2}}.$$

Example 2. Differentiate the function

$$u = \tan^{-1}\frac{2x+1}{3}.$$

Here,

$$du = \frac{d\,\frac{2x+1}{3}}{1+\left(\frac{2x+1}{3}\right)^2} = \frac{\frac{2}{3}\,dx}{\frac{10+4x+4x^2}{9}} = \frac{3\,dx}{5+2x+2x^2},$$

or

$$\frac{du}{dx} = \frac{3}{5+2x+2x^2}.$$

The student should notice that the method used in the text for deriving the fundamental formulas of differentiation is not to be repeated in the applications. It is these formulas themselves that should be used. Thus, to solve Ex. 1 by writing

$$\cos u = \frac{x}{a}$$

and then differentiating would be logically irreproachable, but bad technique.

EXERCISES

Differentiate each of the following functions.

1. $u = \sin^{-1}\frac{x}{a}$. $\qquad \frac{du}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$, if $a > 0$.

2. $u = \tan^{-1}\frac{x}{a}$. $\qquad \frac{du}{dx} = \frac{1}{a^2 + x^2}$.

3. $u = \cot^{-1}\frac{x}{a}$. $\qquad du = -\frac{dx}{a^2 + x^2}$.

4. $u = \sin^{-1}(n \sin x)$. $\qquad \frac{du}{dx} = \frac{n \cos x}{\sqrt{1 - n^2 \sin^2 x}}$.

5. $u = \cos^{-1}\frac{1 - x}{2}$. $\qquad \frac{du}{dx} = \frac{1}{\sqrt{3 + 2x - x^2}}$.

6. $u = \sin^{-1}\frac{2x - 1}{\sqrt{2}}$. $\qquad$ 7. $u = \cot^{-1}\frac{x + a}{b}$.

8. $u = \cot^{-1}\frac{1}{x}$. $\qquad$ 9. $u = \tan^{-1}\frac{a}{x}$.

10. $u = \tan^{-1}\frac{2x}{1 - x^2}$. $\qquad \frac{du}{dx} = \frac{2}{1 + x^2}$.

11. $u = \tan^{-1}\left(x\frac{3 - x^2}{1 - 3x^2}\right)$. $\qquad \frac{du}{dx} = \frac{3}{1 + x^2}$.

12. $u = \sin^{-1}\frac{x - a}{x}$. $\qquad$ 13. $u = \cos^{-1}\frac{x}{x + a}$.

14. $u = \sin^{-1}(2x\sqrt{1 - x^2})$. $\qquad \frac{du}{dx} = \frac{2}{\sqrt{1 - x^2}}$, $|x| < \frac{1}{\sqrt{2}}$

15. $t = \cos^{-1}\frac{s}{2}$. $\qquad \frac{ds}{dt} = -2\sin 2t$.

16. $t = \sin^{-1}\frac{s}{3}$. $\qquad \frac{ds}{dt} = 3\cos 3t$.

17. $t = \cos^{-1}\frac{s}{n} + \gamma$. $\qquad \frac{ds}{dt} = -n\sin n(t-\gamma)$.

18. $u = x\sin^{-1}x$. $\qquad$ 19. $u = \frac{\tan^{-1}x}{x}$.

20. $u = \frac{1}{\sin^{-1}x}$. $\qquad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}(\sin^{-1}x)^2}$.

21. $u = a\cos^{-1}\frac{x-a}{a}$. $\qquad$ 22. $u = \tan^{-1}\frac{x-a}{x+a}$.

23. $u = \cot^{-1}\frac{x+ab}{bx-a}$. $\qquad$ 24 $u = \sin^{-1}\frac{ax+b}{bx+a}$.

25. $u = \sqrt{x^2-a^2} - a\cos^{-1}\frac{a}{x}$. $\qquad \frac{du}{dx} = \frac{\sqrt{x^2-a^2}}{x}, a > 0$.

26. $u = \sin^{-1}\frac{x}{a} + \frac{\sqrt{a^2-x^2}}{x}$. $\qquad \frac{du}{dx} = \frac{\sqrt{a^2-x^2}}{x^2}, a > 0$.

27. $u = \tan^{-1}\left(2\tan\frac{x}{2}\right)$. $\qquad \frac{du}{dx} = \frac{2}{5-3\cos x}$.

28. $u = \tan^{-1}(3\tan\theta)$. $\qquad \frac{du}{d\theta} = \frac{3}{5-4\cos 2\theta}$.

4. Continuation. Numerical Computation. By means of the Tables the numerical value of any of the functions of this chapter can be determined when a specific numerical value has been chosen for the independent variable. It is, however, an important aid to ease and security in such computations to be able, in advance, to make sure of the early significant figures and the location of the decimal point. There are two important geometrical methods for achieving this end. One is the representation of the trigonometric functions by suitable lines connected with the unit circle; the other consists in the graphs introduced above, in § 2.

First of all, however, it should be pointed out that there are two distinct problems. One is to find *all* values of x which satisfy such equations as

(a) $\sin x = .2318;$

(b) $\cos x = -.4322;$

(c) $\tan x = -1.4861.$

The other is to find the *principal value* of an inverse trigonometric function; for example,

$$\sin^{-1}.2318; \qquad \cos^{-1}(-.4322); \qquad \tan^{-1}(-1.4861)$$

The methods of treating these problems are identical.

First Geometric Method. Equations (a), (b), (c) can be solved graphically by the aid of the unit circle representation with an error corresponding to a degree or two, the results being expressed in radians if the problem comes from the Calculus.

For example, consider equation (b). The student should provide himself with an accurately drawn circle of his own construction, executed on the accurate centimeter-millimeter paper commercially procurable; the radius of the circle being 10 cm. and its center at a principal intersection of the rulings.

To solve equation (b), he will lay a straight-edge on his plate, parallel to the secondary (or y-) axis and at a distance of 4 cm., $3\frac{1}{4}$ mm. to the left of that axis. Marking the two points of intersection of the straight-edge with the circle by fine pencil lines easily erased, he now measures one of the acute angles involved by means of his protractor and thus determines the two solutions of (b) lying between 0° and 360° correct to minutes or thereabouts. By aid of the Tables the values can at once be converted into radian measure.

Arithmetic Solutions. From the figure before him the student now sees clearly a right triangle, one leg of which is known. The determination of the angle he needs is merely a problem in the solution of a right triangle by the tables, and

he proceeds to carry this work through to the degree of accuracy which the tables permit.

Equations (a) and (c) are treated in a similar manner. The point of this method is that the student is trained to *visualize a figure*, and not to try to remember a table that looks like

$$\sin A \qquad + \qquad + \qquad - \qquad -.$$

For, such tables vanish in a short time, and when the student needs his trigonometry in later work, he is helpless.

In terms of the inverse functions, this first problem consists in finding all the values of the multiple-valued function $\cos^{-1} x$ for the value of the variable, $x = -.4322$.

Second Geometric Method. This method consists in reading off from the graph the two values which lie between 0 and 2π, and then adding to these arbitrary positive or negative multiples of 2π.

The graph suggests, moreover, how to determine these values arithmetically by the aid of a table of sines or cosines of angles of the first quadrant. It also suggests a further refinement of the graphical method, of which the student will do well to avail himself, — namely, this. Let him make an accurate graph of the function

$$y = \sin x$$

on cm.-mm.-paper, taking 10 cm. as the unit and measuring the angle in radians, x ranging from 0 to $\pi/2$. This half-arch supplements the four graphs of the functions $\sin x$, $\cos x$, $\sin^{-1} x$, $\cos^{-1} x$ and serves as a 3-place table for determining their values (with a possible error of two or three units in the third place).

To sum up, then, there are two geometric methods; 1) the unit-circle method; 2) the graphs of the functions, the latter being supplemented by the 10-cm. graph just described. Either of the geometric methods suggests how to use the tables correctly and affords an altogether satisfactory check on the tables.

When the accurately drawn graphs are not at hand, freehand drawings indicate clearly how to use the tables with security and accuracy.

EXERCISES

1. Determine both in degrees and radians all values of x which satisfy the above equations (a), (b), (c), using each time all of the geometric methods set forth, and also the tables.

2. Find the value of each of the following functions. It is understood that the *principal value* is meant. Use first the method of the graphs. Then determine from the tables. Check by unit-circle and protractor.

i) $\sin^{-1}(-.1643)$; ii) $\cos^{-1}(.6417)$;

iii) $\tan^{-1}(-2.8162)$.

3. By means of a free-hand drawing of the graph estimate the value of each of the following functions. Remember that a curve recedes from its tangent very slowly near a point of inflection.

a) $\sin^{-1}.113$;	b) $\tan^{-1}(-.214)$;	c) $\cos^{-1}.172$;
d) $\tan^{-1}(-7.4)$;	e) $\cot^{-1}(-.152)$;	f) $\cos^{-1}(-.998)$;
g) $\sin^{-1}(-.21)$;	h) $\sin^{-1}.89$;	i) $\tan^{-1}5\,2$;
j) $\cot^{-1}7.3$;	k) $\cos^{-1}(-.138)$;	l) $\sin^{-1}(-.138)$.

In what cases is your error large; in what, small?

5. Applications. The inverse trigonometric functions afford a convenient means of solving the following problem in Optics.

A ray of light is refracted in a prism. Show that its deviation from its original direction is least when the incident ray and the refracted ray make equal angles with the faces of the prism.

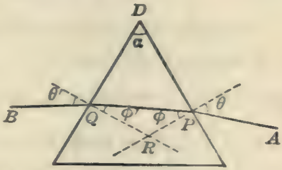


FIG. 73

The study of this problem has a vivid interest for the student who has seen the laboratory experiment of admitting a ray of sunlight into a darkened room, allowing it to pass through

a prism, thus being refracted, and throwing it finally, dispersed, on a screen.

Let AP be the incident ray; PQ, its path through the prism; and QB the ray which emerges. Then the deflection of PQ is obviously $\theta - \phi$ and the further deflection of QB is $\theta' - \phi'$; so that the total deflection, u, is:

(1) $$u = \theta - \phi + \theta' - \phi' = \theta + \theta' - (\phi + \phi').$$

On the other hand, the sum of the angles of the triangle PDQ is

$$\pi = \left(\frac{\pi}{2} - \phi\right) + \left(\frac{\pi}{2} - \phi'\right) + \alpha.$$

Hence

(2) $$\phi + \phi' = \alpha.$$

We can, therefore, write (1) in the form:

(3) $$u = \theta + \theta' - \alpha.$$

This is the quantity it is desired to make a minimum. θ and θ' are, however, connected by a relation which can be obtained as follows. We have by the law of refraction (cf. Chap. V, § 7):

(4) $$\frac{\sin\theta}{\sin\phi} = n, \qquad \frac{\sin\theta'}{\sin\phi'} = n.$$

Let $\nu = 1/n$. Then

(5) $$\sin\phi = \nu\sin\theta \qquad \text{or} \qquad \phi = \sin^{-1}(\nu\sin\theta).$$

Similarly,

(6) $$\sin\phi' = \nu\sin\theta' \qquad \text{or} \qquad \phi' = \sin^{-1}(\nu\sin\theta').$$

Substituting these values of ϕ and ϕ' in equation (2) we have the desired relation:

(7) $$\sin^{-1}(\nu\sin\theta) + \sin^{-1}(\nu\sin\theta') = \alpha.$$

Our problem now is completely formulated; it is: To make the function u given by (3) a minimum, when θ and θ' are connected by (7):

(8) $$\begin{cases} u = \theta + \theta' - \alpha, \\ \sin^{-1}(\nu\sin\theta) + \sin^{-1}(\nu\sin\theta') = \alpha. \end{cases}$$

Take θ as the independent variable. Then

(9) $$\frac{du}{d\theta}=1+\frac{d\theta'}{d\theta},$$

and the condition

$$\frac{du}{d\theta}=0 \qquad \text{gives} \qquad \frac{d\theta'}{d\theta}=-1.$$

Next, take the differential of each side of the second equation (8):

$$\frac{d(\nu\sin\theta)}{\sqrt{1-\nu^2\sin^2\theta}}+\frac{d(\nu\sin\theta')}{\sqrt{1-\nu^2\sin^2\theta'}}=0,$$

or

$$\frac{\nu\cos\theta\,d\theta}{\sqrt{1-\nu^2\sin^2\theta}}+\frac{\nu\cos\theta'\,d\theta'}{\sqrt{1-\nu^2\sin^2\theta'}}=0.$$

Hence

(10) $$\frac{\cos\theta}{\sqrt{1-\nu^2\sin^2\theta}}+\frac{\cos\theta'}{\sqrt{1-\nu^2\sin^2\theta'}}\left(\frac{d\theta'}{d\theta}\right)=0.$$

But $d\theta'/d\theta=-1$. Consequently

(11) $$\frac{\cos\theta}{\sqrt{1-\nu^2\sin^2\theta}}=\frac{\cos\theta'}{\sqrt{1-\nu^2\sin^2\theta'}}.$$

One solution of this equation is $\theta=\theta'$, — the solution demanded by the theorem. But conceivably there might be other solutions, and then it would not be clear which one of them makes u a minimum. We can readily show, however, that equation (11) has no further solutions. Square each side:

$$\frac{\cos^2\theta}{1-\nu^2\sin^2\theta}=\frac{\cos^2\theta'}{1-\nu^2\sin^2\theta'}.$$

Clear of fractions and express each cosine in terms of the sine:

$$(1-\nu^2\sin^2\theta')(1-\sin^2\theta)=(1-\nu^2\sin^2\theta)(1-\sin^2\theta').$$

Multiply out and suppress equal terms on the two sides:

$$-\sin^2\theta-\nu^2\sin^2\theta'=-\sin^2\theta'-\nu^2\sin^2\theta,$$

$$(\nu^2-1)\sin^2\theta=(\nu^2-1)\sin^2\theta'.$$

Hence

$$\sin^2\theta = \sin^2\theta', \qquad \sin\theta = \sin\theta',$$

and consequently the only angles of the first quadrant which can satisfy (11) are equal angles, $\theta = \theta'$.

From (5) and (6) it follows that $\phi = \phi'$. Hence, from (2)

$$\phi = \frac{\alpha}{2}, \qquad \text{and so} \qquad \theta = \sin^{-1}\left(n \sin\frac{\alpha}{2}\right),$$

$$u = 2\sin^{-1}\left(n \sin\frac{\alpha}{2}\right) - \alpha.$$

That u is a minimum, is clearly indicated by the laboratory experiment. It can be proven analytically as follows. From (9)

$$\frac{d^2u}{d\theta^2} = \frac{d^2\theta'}{d\theta^2}.$$

Differentiate (10) as it stands; then, after the differentiation, set $d\theta'/d\theta = -1$ and $\theta = \theta'$. It is seen at once that

$$\frac{d^2\theta'}{d\theta^2} > 0, \qquad \text{hence} \qquad \frac{d^2u}{d\theta^2} > 0,$$

and u has a minimum.

EXERCISE

The bottom of a mural painting 4 ft. high is 12 ft. above the eye of the observer. How far back from the wall should he stand, in order that the angle subtended by the painting be as large as possible?

Suggestion. Take the distance, x, of the observer from the wall as the independent variable, and express the angle of elevation of the bottom and the top of the painting in terms of x.

CHAPTER IX

INTEGRATION

1. The Area under a Curve. Let it be required to compute the area bounded by the curve

$$(1) \qquad y = f(x),$$

the axis of x, and two ordinates whose abscissas are $x = a$ and $x = b$, $(a < b)$, Fig. 74. We can proceed as follows. Consider first the variable area, A, bounded by the first three lines just mentioned and an ordinate whose abscissa x is variable. Then A is a function of x. For, when we assign to x any value between the limits a and b in question, the corresponding value of the area is thereby determined and could actually be computed by plotting the figure on squared paper and counting the squares, or by cutting the figure out of a sheet of paper or tin and weighing the piece.

If, then, we can obtain an analytic expression for this function of x, holding for all values of x from a to b, we can then set $x = b$ in this formula and thus solve the above problem.

To do this, begin by giving to x an arbitrary value, $x = x_0$, and denoting the corresponding value of A by A_0. Next, give to x an increment, Δx, and denote the corresponding increment in A by ΔA. We can approximate to the area ΔA

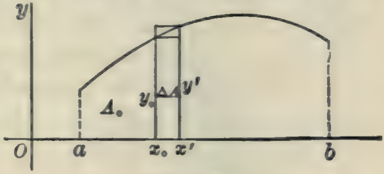


FIG. 74

by means of two rectangles, as shown in the figure, — an inscribed rectangle, whose area is $y_0 \Delta x$, and a circumscribing rectangle, whose area is $(y_0 + \Delta y)\Delta x$, — and thus we get:

$$y_0 \Delta x < \Delta A < (y_0 + \Delta y)\Delta x.$$

Hence
$$y_0 < \frac{\Delta A}{\Delta x} < y_0 + \Delta y.$$

Now allow Δx to approach 0 as its limit. The variable $\Delta A/\Delta x$ always lies between the fixed number y_0 and the variable $y_0 + \Delta y$ which is approaching y_0 as its limit. It follows, then, that $\Delta A/\Delta x$ must also approach y_0:

$$\lim_{\Delta x \doteq 0} \frac{\Delta A}{\Delta x} = y_0.$$

The limit on the left-hand side is, by definition, $[D_x A]_{x=x_0}$. Hence, dropping the subscript, which has now served its purpose, we have:

(2) $$D_x A = y,$$

or

(2′) $$D_x A = f(x).$$

If $f(x)$ is decreasing as x passes through the value x_0, the signs of inequality must point the other way:

$$y_0 \Delta x > \Delta A > (y_0 + \Delta y)\Delta x,$$

etc. The reasoning is, however, essentially the same, and the result, namely, equation (2), is identical.

The answer to our question: What is A? thus comes to us in the form of a riddle. Tell me what function must be differentiated to yield the given function (1), *i.e.*, $f(x)$, and I will tell you the area.

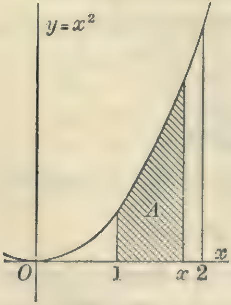


Fig. 75

The riddle can be easily answered in simple cases. Suppose the curve (1) is the parabola

(3) $$y = x^2,$$

and let $a = 1$, $b = 2$.* Here,

$$f(x) = x^2,$$

and (2) or (2′) becomes

$$D_x A = x^2.$$

* The parabola is drawn accurately in Fig. 75; but the ordinates and the area are schematic, *i.e.* not drawn to scale.

The question now is: What function must we differentiate in order to get x^2? We readily see that $x^3/3$ is such a function. But this is not the only one. For, if we add any constant, $x^3/3 + C$ will also differentiate into x^2. We shall see later that this is the most general function whose derivative is x^2, and hence A must be of the form:

$$(4) \qquad A = \frac{x^3}{3} + C.$$

This formula is not wholly definite, for C may be any constant. On the other hand we have not as yet brought all our data into play, for we have as yet said nothing about the fact that the left-hand ordinate shall correspond to the abscissa $x=1$. Now the variable area A will be small when x is only a little greater than 1, and it will approach 0 as its limit when x approaches 1. If, then, (4) is to be a true formula, it must give 0 as the value of A when $x = 1$, or

$$(5) \qquad 0 = \tfrac{1}{3} + C, \qquad C = -\tfrac{1}{3}.$$

$$(6) \qquad \therefore\ A = \frac{x^3}{3} - \frac{1}{3}.$$

Having thus found the variable area, we can now obtain the area we set out to compute by putting $x = 2$ in (6):

$$[A]_{x=2} = \tfrac{8}{3} - \tfrac{1}{3} = 2\tfrac{1}{3}.$$

The process of finding the area under a curve is thus seen to be as follows. First find a function which, when differentiated, will give the ordinate $y = f(x)$ of the curve (1) before us; and add an undetermined constant to this function. Next. determine this constant by requiring that A shall $= 0$ when $x = a$. Thus the variable area is completely expressed as a function of x. Lastly, substitute $x = b$ in this formula.

EXERCISES

1. Show that, if the variable area in the foregoing example had been measured from the fixed ordinate $x = 2$, the value of the constant C would have been $-2\tfrac{2}{3}$:

$$A = \frac{x^3}{3} - 2\tfrac{2}{3};$$

and if it had been measured from the origin, then C would have been $= 0$:

$$A = \frac{x^3}{3}.$$

2. If, in (1), $y = f(x) = x$, the curve is a straight line; and if $a = 6$, $b = 20$, the figure is a trapezoid. Compute its area by the above method and check your result by elementary geometry.

3. Find the area under the curve

$$y = x^4,$$

lying between the ordinates whose abscissas are $x = 10$ and $x = 20$. *Ans.* 620,000.

4. Find the area of one arch of the curve

$$y = \sin x.$$ *Ans.* 2.

5. Find the area under that portion of the curve

$$y = 1 - x^2$$

which lies above the axis of x.

6. A river bends around a meadow, making a curve that is approximately a parabola:

$$y = x - 4x^2,$$

referred to a straight road that crosses the river, as axis of x; the mile is taken as the unit. How many acres of meadow are there between the road and the river? *Ans.* $6\frac{2}{3}$, nearly.

2. **The Integral.** In the preceding chapters we have treated the problem: Given a function; to find its derivative. The examples of the last paragraph are typical for the inverse problem: Given the derivative of a function; to find the function. Stated in equations, the problem is this. If

$$D_x U = u, \qquad \text{or} \qquad dU = u\,dx,$$

where u is given, to find U.

The function U is called the *integral* of u with respect to x and is denoted as follows:

$$U = \int u\,dx$$

Thus we have the following

Definition of an Integral. The function U is said to be the integral of u with respect to x:

$$U = \int u\,dx,$$

if $\qquad D_xU = u, \qquad$ or $\qquad dU = u\,dx.$

The given function u is called the *integrand.*

For example,

(1) $$\int x^5\,dx = \frac{x^6}{6} + C.$$

For, if we set

$$U = \frac{x^6}{6} + C$$

and differentiate this function with respect to x, we get:

$$D_x U = x^5.$$

This last function is precisely the integrand,

$$u = x^5.$$

Hence equation (1) is true by definition.

It follows from the definition that *differentiation and integration are inverse processes.* One undoes what the other does. Just as

$$\sqrt[3]{x^3} = x, \qquad \text{or} \qquad \log e^x = x, \qquad \text{or} \qquad \sin(\sin^{-1} x) = x,$$

so

(2) $$D_x \int u\,dx = u.$$

Thus we can test conveniently any formula of integration by differentiating back. For example, is

(3) $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

a true equation? Differentiate each side:

$$D_x\int x^n\,dx = D_x\left(\frac{x^{n+1}}{n+1}+C\right).$$

The value of the left-hand side is, by (2), $u = x^n$. The value of the right-hand side is $D_x U = x^n$. Since these two functions are the same, *i.e.*, since $D_x U = u$, it follows from the definition that (3) is true.

EXERCISES

1. Show that

$$\int \sin x\,dx = -\cos x + C.$$

Solution. Differentiate each side:

$$D_x\int \sin x\,dx = D_x(-\cos x + C).$$

If this is a true equation, then the given equation is true. Now

$$D_x\int \sin x\,dx = \sin x$$

and $$D_x(-\cos x + C) = \sin x.$$

Hence the given equation is true.

Prove the following equations to be correct.

2. $\displaystyle\int \cos x\,dx = \sin x + C.$

3. $\displaystyle\int \sec^2 x\,dx = \tan x + C.$

4. $\displaystyle\int \frac{dx}{x} = \log x + C.$

5. $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1} x + C.$

6. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$

7. $\displaystyle\int 0\,dx = C.$

Evaluate the following integrals.

8. $\displaystyle\int \sqrt{x}\,dx.$ *Ans.* $\frac{2}{3}x^{\frac{3}{2}} + C.$

9. $\displaystyle\int \frac{dx}{\sqrt{x}}.$ *Ans.* $2\sqrt{x} + C.$

10. $\displaystyle\int \frac{dx}{x^2}$. *Ans.* $-\frac{1}{x}+C$.

11. $\displaystyle\int x^{\frac{2}{3}}\,dx$. *Ans.* $\frac{3}{5}x^{\frac{5}{3}}+C$.

12. $\displaystyle\int \frac{dx}{x^3}$. 13. $\displaystyle\int \frac{dx}{x^4}$. 14. $\displaystyle\int \frac{dx}{x^5}$.

3. General Theorems. We will first show that all the integrals of a given function are obtained from a particular integral by adding an arbitrary constant to the latter. Suppose, then, that U and U' are any two integrals of u with respect to x:

$$U=\int u\,dx \qquad \text{and} \qquad U'=\int u\,dx.$$

Then, by definition:

$$D_xU=u \qquad \text{and} \qquad D_xU'=u.$$

From these two equations we infer that

$$D_xU'-D_xU=0, \qquad \text{or} \qquad D_x(U'-U)=0.$$

Hence $U'-U$ is a function, $\phi(x)$, whose derivative is always 0. Plot its graph:

$$y=\phi(x).$$

The slope of this curve, D_xy, is always 0. Hence the curve must be a straight line parallel to the axis of x, and the equation of such a line is

$$y=C.$$

For, if the graph were any other curve, there would be points near which the ordinate is increasing as x increases, or else points near which the ordinate is decreasing as x increases. In either case, there would result points of the graph at which the slope is not 0.

We see, then, that $\phi(x)=C$; hence

$$U'-U=C, \qquad \text{or} \qquad U'=U+C,$$

and this is what we set out to prove.

The first two General Theorems of Differentiation, Chap. II, § 6, pp. 29, 30, find their counterpart in integration.

THEOREM I. *A constant factor can be taken out from under the sign of integration:*

$$\text{(I)} \qquad \int c\,u\,dx = c\int u\,dx.$$

Thus, when we have once seen that

$$\int \sin x\,dx = -\cos x + C,$$

it follows by this theorem that

$$\int 2\sin x\,dx = -2\cos x + C'.$$

Likewise,

$$\int cx^n\,dx = \frac{cx^{n+1}}{n+1} + C, \qquad n \neq -1.$$

But a *variable* factor cannot be taken out from under the integral sign. For example, it is not true that

$$\int x\sin x\,dx = x\int \sin x\,dx = x(-\cos x + C).$$

For,

$$D_x[x(-\cos x + C)] = x\sin x - \cos x + C,$$

and this function is not the same as the integrand, $x\sin x$.

Proof of the Theorem. Form the function:

$$\phi(x) = \int c\,u\,dx - c\int u\,dx,$$

and differentiate it:

$$D_x\phi(x) = D_x\int c\,u\,dx - cD_x\int u\,dx$$
$$= cu - cu = 0.$$

Thus the derivative of $\phi(x)$ is always 0. Consequently $\phi(x)$, as we have seen above, must be a constant, C. Hence we infer that

$$\text{(1)} \qquad \int c\,u\,dx = c\int u\,dx + C.$$

This does not appear to be precisely what we set out to prove, since the term C is absent in (I). But obviously we cannot choose *both* integrals in (I) arbitrarily. What (I) means is that, when we have chosen $\int u\,dx$ to be any *particular* integral of u with respect to x, then *one* of the integrals $\int cu\,dx$ is $c\int u\,dx$, and this result is contained in (1).

THEOREM II. *The integral of the sum of two functions is equal to the sum of their integrals:*

$$\text{(II)} \qquad \int (u+v)\,dx = \int u\,dx + \int v\,dx.$$

Proof. Form the function:

$$\phi(x) = \int (u+v)dx - \int u\,dx - \int v\,dx,$$

and differentiate it:

$$D_x\phi(x) = D_x\int (u+v)\,dx - D_x\int u\,dx - D_x\int v\,dx$$
$$= u + v - u - v = 0.$$

Hence $\phi(x)$ is a constant, C, and we have

$$\int (u+v)\,dx = \int u\,dx + \int v\,dx + C.$$

This is the equation which holds when all three integrals are chosen arbitrarily. If any two are chosen arbitrarily, then a particular value of the third can be found which will make $C = 0$. Hence the theorem is proved.

Application. By means of these two theorems any polynomial can be integrated. Thus

$$\int (3x^2 - x + 1)\,dx = \int 3x^2\,dx + \int -x\,dx + \int dx$$
$$= 3\frac{x^3}{3} + (-1)\frac{x^2}{2} + x + C$$
$$= x^3 - \tfrac{1}{2}x^2 + x + C.$$

It would have been allowable to add a constant to each integral that was evaluated. But the result would not have been more general, since the sum of three arbitrary constants is merely an arbitrary constant.

Area under a Curve. We can now express the area discussed in § 1 in the form:

$$(1) \qquad A = \int y\,dx \qquad \text{or} \qquad A = \int f(x)\,dx.$$

In § 1 it was tacitly assumed that the function $f(x)$ is positive. If $f(x)$ is negative, so that the curve lies below the axis, we will agree to consider the area bounded by the curve as negative, and then we arrive at the same formula, $D_x A = y$, as before.

If, finally, part of the curve lies above the axis of x and part below, we will regard the total area under the curve as the algebraic sum of the part above the axis, reckoned positive, and the part below the axis, reckoned negative.

EXERCISES

Evaluate the following integrals.

1. $\int (x^3 + x - 1)\,dx.$ *Ans.* $\frac{1}{4}x^4 + \frac{1}{2}x^2 - x + C.$

2. $\int \frac{x^2 + x + 1}{3}\,dx.$ *Ans.* $\frac{1}{9}x^3 + \frac{1}{6}x^2 + \frac{1}{3}x + C.$

3. $\int (3 - 4x - 9x^8)\,dx.$ *Ans.* $3x - 2x^2 - x^9 + C.$

4. $\int (\sqrt{2}\,x + \pi)\,dx.$ *Ans.* $\frac{x^2}{\sqrt{2}} + \pi x + C.$

5. $\int (v_0 - gt)\,dt.$ *Ans.* $v_0 t - \frac{1}{2}gt^2 + C.$

6. $\int \frac{1 + x}{\sqrt{x}}\,dx.$ *Ans.* $2\sqrt{x} + \frac{2}{3}\sqrt{x^3} + C.$

7. $\int \frac{1 + x}{\sqrt[3]{x}}\,dx.$ 8. $\int (x^{\frac{2}{3}} - x^{-\frac{1}{3}})\,dx.$ 9. $\int \frac{1 - x^2}{\sqrt{x}}\,dx.$

10. $\displaystyle\int\frac{1+x+x^2}{x}\,dx.$ *Ans.* $\log x + x + \frac{1}{2}x^2 + C.$

11. $\displaystyle\int\frac{1+x^4}{x^2}\,dx.$ 12. $\displaystyle\int\frac{2-3x+6x^2}{x^2}\,dx.$

13. Find the area above the positive axis of x bounded by the curve:

$$y=(x^2-1)(4-x^2).$$ *Ans.* $1\frac{7}{15}$.

14. Find the area enclosed between the two parabolas:

$$y=x^2, \qquad y^2=x.$$

4. Special Formulas of Integration. Corresponding to the Special Formulas of Differentiation, p. 215, we can write down a list of special formulas of integration, by means of which, together with the general methods discussed in this chapter, all the simpler integrals can be evaluated. Each formula can be proven by differentiating each side of the equation, as explained in § 2.

The constant of integration is omitted for brevity, just as, in a table of logarithms, the decimal point is omitted. But in applying the formulas the constant must always be included.

Special Formulas of Integration

1. $$\int x^n\,dx=\frac{x^{n+1}}{n+1}, \qquad n\neq -1.$$

2. $$\int \sin x\,dx = -\cos x.$$

3. $$\int \cos x\,dx = \sin x.$$

4. $$\int \frac{dx}{x} = \log x.$$

5. $$\int e^x\,dx = e^x.$$

6. $$\int \frac{dx}{1+x^2} = \tan^{-1} x.$$

7. $$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x,$$
$$= - \cos^{-1} x.$$

8. $$\int \sec^2 x \, dx = \tan x.$$

9. $$\int \csc^2 x \, dx = - \cot x.$$

To these may be added the formulas:

10. $$\int \frac{dx}{\sqrt{2x - x^2}} = \text{vers}^{-1} x.$$

11. $$\int a^x \, dx = \frac{a^x}{\log a}.$$

The formula

$$\int 0 \, dx = C$$

should also be added for the sake of completeness.

5. **Integration by Substitution.** Many integrals can be obtained from the special formulas of § 4 by introducing a new variable of integration. The following examples will illustrate the method.

Example 1. To find $\int \sqrt{a+bx}\, dx.$

Let $a + bx = y.$ Then $b\,dx = dy$

and $$\sqrt{a+bx}\, dx = \frac{1}{b} y^{\frac{1}{2}} dy.$$

Integrating each side of this equation, we get:

$$\int \sqrt{a+bx}\, dx = \frac{1}{b}\int y^{\frac{1}{2}} dy = \frac{1}{b}\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + C.$$

Hence $$\int \sqrt{a+bx}\, dx = \frac{2\sqrt{(a+bx)^3}}{3b} + C.$$

Example 2. To find $\int \cos ax\, dx.$

Let $\quad ax = y. \quad$ Then $\quad a\,dx = dy,$

$$\cos ax\, dx = \frac{1}{a}\cos y\, dy,$$

and

$$\int \cos ax\, dx = \frac{1}{a}\int \cos y\, dy = \frac{1}{a}\sin y + C$$
$$= \frac{1}{a}\sin ax + C.$$

Example 3. To find $\int x\sqrt{a^2 + x^2}\, dx.$

Let $\quad x^2 = y. \quad$ Then $\quad 2\,x\,dx = dy,$

$$x\sqrt{a^2 + x^2}\, dx = x\sqrt{a^2 + y}\,\frac{dy}{2\,x} = \tfrac{1}{2}\sqrt{a^2 + y}\, dy,$$

and

$$\int x\sqrt{a^2 + x^2}\, dx = \tfrac{1}{2}\int \sqrt{a^2 + y}\, dy.$$

This last integral is a special case of the integral of Example 1. For, if the a of that formula is replaced by a^2, the b by 1, and the x by y, we have the present integral. Hence

$$\int x\sqrt{a^2 + x^2}\, dx = \tfrac{1}{3}(a^2 + x^2)^{\frac{3}{2}} + C.$$

We might have set $a^2 + x^2 = y$.

Example 4. To find $\int \tan x\, dx.$

Here

$$\int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x} = \int \frac{-\,d\cos x}{\cos x}$$
$$= -\log\cos x + C.$$

In substance, we have introduced a new variable, $\cos x = y$. But in practice it is often simpler, as here, to refrain from actually writing a new letter.

In the above examples we have tacitly assumed that if x and

y are functions one of the other, and if $f(x)$ and $\phi(y)$ are two functions such that

$$(1) \qquad f(x)\,dx = \phi(y)\,dy,$$

then

$$(2) \qquad \int f(x)\,dx = \int \phi(y)\,dy.$$

We can justify this assumption without difficulty. For,

$$D_x \int f(x)\,dx = f(x);$$

$$D_x \int \phi(y)\,dy = D_y \int \phi(y)\,dy \cdot D_x y = \phi(y)\,D_x y;$$

and since, by (1), $\qquad f(x) = \phi(y)\dfrac{dy}{dx},$

we see that the two integrals in equation (2) have equal derivatives. It follows, then, from § 3 that the above integrals differ from each other at most by a constant, C. Hence, if the constant of integration in one of these integrals is chosen at pleasure, the constant of integration in the other can be so determined that $C = 0$.

This theorem in integration corresponds to Theorem V of Chap. II in differentiation. As in the case of that theorem, the use of differentials, — and it is to this fact that their importance is due, — reduces the theorem in form to an identity:

$$\int u\,dx = \int \left[u\frac{dx}{dy}\right] dy.$$

EXERCISES

Evaluate the following integrals*: —

1. $\displaystyle\int \sqrt{1-x}\,dx.$ *Ans.* $-\frac{2}{3}(1-x)^{\frac{3}{2}} + C.$

* The student should take care not to be unduly delayed by a particular integral in the following list which may give him difficulty. He should work first those examples which seem easiest, and then note precisely why

2. $\int \sqrt[3]{1+2x}\,dx$. *Ans.* $\frac{3}{8}(1+2x)^{\frac{4}{3}} + C$.

3. $\int \frac{dx}{\sqrt{3-2x}}$. 4. $\int \frac{dx}{\sqrt[4]{a+bx}}$. 5. $\int (a+bx)^n\,dx$.

6. $\int \sin ax\,dx$. 7. $\int \cos\frac{x}{2}\,dx$. 8. $\int \sin(\pi x+\gamma)\,dx$.

9. $\int \frac{dx}{a^2+x^2}$. *Ans.* $\frac{1}{a}\tan^{-1}\frac{x}{a} + C$.

10. $\int \frac{dx}{\sqrt{a^2-x^2}}$. *Ans.* $\sin^{-1}\frac{x}{a} + C$.

11. $\int \frac{x\,dx}{\sqrt{a^2-x^2}}$. *Ans.* $-\sqrt{a^2-x^2} + C$.

12. $\int \frac{x\,dx}{\sqrt{x^2-a^2}}$. 13. $\int \frac{x\,dx}{\sqrt{a^2+x^2}}$. 14. $\int x\sqrt{x^2-a^2}\,dx$.

15. $\int x^2\sqrt{a^3+x^3}\,dx$. 16. $\int x\sin x^2\,dx$. 17. $\int e^{-x}\,dx$.

18. $\int \frac{dx}{a+bx}$. 19. $\int \frac{x\,dx}{a+bx^2}$. 20. $\int \frac{dx}{(1-x)^2}$.

21. $\int \cot x\,dx$. 22. $\int \sin 2x\,dx$. 23. $\int \cos 3x\,dx$.

24. $\int (e^{-x})^2\,dx$. 25. $\int \tan ax\,dx$. 26. $\int \cot 2x\,dx$.

27. $\int \frac{2\,dx}{2-3x}$. 28. $\int \frac{dx}{(2-3x)^2}$. 29. $\int \frac{5\,dx}{(7-4x)^{\frac{2}{3}}}$.

30. $\int \frac{2x\,dx}{1+x^2}$. 31. $\int \frac{x\,dx}{x^2-4}$. 32. $\int \frac{x^2\,dx}{a^3+x^3}$.

the others gave trouble. It is well to copy off the latter on separate sheets of paper, lay them aside for a few days, and then work them again.

It is a question here rather of observational science than of logic. A class-mate of mine, Mr. W. B. Stone, aptly compared this work to that of analyzing a flower in botany.

33. $\int x\, e^{-x^2}\, dx.$ 34. $\int e^{ax}\, dx.$ 35. $\int \tan\frac{x}{2}\, dx.$

36. $\int (a^2 - x^2)^2\, dx.$ 37. $\int (e^x - e^{-x})^2\, dx.$ 38. $\int (a^x + a^{-x})^2\, dx$

6. Integration by Ingenious Devices.

Example 1. To find $\int \cos^2\theta\, d\theta.$

Set $$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta).$$

Then $$\int \cos^2\theta\, d\theta = \tfrac{1}{2}\int (1 + \cos 2\theta)\, d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C,$$

(1) $$\therefore \int \cos^2\theta\, d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C.$$

We can now evaluate an important integral, namely:

$$\int \sqrt{a^2 - x^2}\, dx.$$

Let $$x = a\sin\theta; \qquad dx = a\cos\theta\, d\theta,$$

$$\sqrt{a^2 - x^2}\, dx = a^2\cos^2\theta\, d\theta,$$

$$\int \sqrt{a^2 - x^2}\, dx = a^2\int \cos^2\theta\, d\theta = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C,$$

(2) $$\therefore \int \sqrt{a^2 - x^2}\, dx = \tfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C.$$

A simple geometric evaluation of this integral can be given by observing that it is represented by the area under the circle

$$x^2 + y^2 = a^2.$$

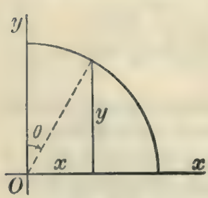


FIG. 76

This area can be cut into two parts, — a triangle and a circular sector, — and the areas of these pieces are precisely the two terms in x on the right-hand side of (2).

Example 2. To find $\int \frac{dx}{a^2 - x^2}$.

The integrand can be written in the form:

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left[\frac{1}{x+a} - \frac{1}{x-a}\right].$$

Hence
$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int \frac{dx}{x+a} - \frac{1}{2a}\int \frac{dx}{x-a}$$
$$= \frac{1}{2a}[\log(x+a) - \log(x-a)] + C;$$

(3)
$$\therefore \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\frac{x+a}{x-a} + C.^*$$

Example 3. To find $\int \frac{d\theta}{\sin\theta}$.

* In case $-a < x < a$, formula (3) involves the logarithm of a negative quantity. We can avoid this difficulty by writing the second term in the bracket as $+1/(a-x)$, the corresponding integral thus becoming

$$\int \frac{dx}{a-x} = -\log(a-x).$$

This leads to the formula:

(3′)
$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\frac{a+x}{a-x} + C.$$

It will be shown later in the Calculus that, in the domain of imaginaries, the logarithms of (3) and (3′) differ from each other only by an imaginary constant, and since the latter may be included in the constant of integration, (3) and (3′) may be regarded as equivalent formulas.

No matter which formula is used in an applied problem, after the constant of integration has been determined, or the integral taken between limits as in Chap. XII, the final result will be the same. Hence it is not necessary, in tabulating integrals whose values involve logarithms, to enter two formulas corresponding to a possible negative function under the sign of the logarithm.

First Method. $$\int \frac{d\theta}{\sin\theta} = \int \frac{d\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}.$$

Let $\frac{\theta}{2} = \phi$. Then the last integral becomes:

$$\int \frac{d\phi}{\sin\phi\cos\phi} = \int \frac{\sec^2\phi\, d\phi}{\tan\phi} = \int \frac{d\tan\phi}{\tan\phi} = \log\tan\phi + C.$$

(4) $$\therefore \int \frac{d\theta}{\sin\theta} = \log\tan\frac{\theta}{2} + C.$$

Second Method.

$$\int \frac{d\theta}{\sin\theta} = \int \frac{\sin\theta\, d\theta}{\sin^2\theta}$$

$$= -\int \frac{d\cos\theta}{1-\cos^2\theta} = -\tfrac{1}{2}\log\frac{1+\cos\theta}{1-\cos\theta} + C.$$

The fraction $$\frac{1+\cos\theta}{1-\cos\theta} = \frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \frac{1}{\tan^2\frac{\theta}{2}}.$$

(4) $$\therefore \int \frac{d\theta}{\sin\theta} = \log\tan\frac{\theta}{2} + C.$$

EXERCISES

1. $\int \sin^2\theta\, d\theta.$ *Ans.* $\frac{1}{2}(\theta - \sin\theta\cos\theta) + C.$

2. $\int \frac{d\theta}{1+\cos\theta}.$

3. $\int \frac{d\theta}{1-\cos\theta}.$

4. $\int \frac{d\theta}{\cos\theta}.$ *Ans.* $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + C,$

or $\frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta} + C,$ or $\log(\sec\theta + \tan\theta) + C.$

5. $\int \frac{dx}{\sqrt{x^2+a^2}}.$ *Ans.* $\log(x + \sqrt{x^2+a^2}) + C.$

Suggestion. Let $x = a\tan\theta$.

6. $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}$. *Ans.* $\log(x+\sqrt{x^2-a^2})+C.$

7. $\displaystyle\int\frac{d\theta}{\sin 2\theta}$. 8. $\displaystyle\int\frac{d\theta}{\cos\frac{1}{2}\theta}$. 9. $\displaystyle\int\frac{d\theta}{\sin 3\theta}$.

10. $\displaystyle\int\frac{d\theta}{1+\sin\theta}$. 11. $\displaystyle\int\frac{d\theta}{1-\sin\theta}$. 12. $\displaystyle\int\frac{d\theta}{1+\sin 2\theta}$.

13. $\displaystyle\int(1+\cos\theta)^2\,d\theta$. 14. $\displaystyle\int(1-2\sin\theta)^2\,d\theta$.

15. $\displaystyle\int(1-\cos 2\theta)^2\,d\theta$. 16. $\displaystyle\int(4+\sin 3\theta)^2\,d\theta$.

7. Integration by Parts.* The formula of differentiation:

$$d(uv)=u\,dv+v\,du,$$

leads to the formula of integration:

$$\int u\,dv=uv-\int v\,du. \tag{1}$$

Integration by means of this formula is known as *Integration by Parts.*

Example 1. To find $\displaystyle\int xe^x\,dx$.

Let $\qquad u=x, \qquad dv=e^x\,dx;$

then $\qquad du=dx, \qquad v=\displaystyle\int e^x\,dx=e^x,$

and $\qquad \displaystyle\int xe^x\,dx=xe^x-\int e^x\,dx=(x-1)e^x+C.$

Example 2. To find $\displaystyle\int\log x\,dx$.

Let $\qquad u=\log x, \qquad dv=dx;$

then $\qquad du=\dfrac{dx}{x}, \qquad v=x,$

* This paragraph should be omitted, or treated only cursorily, in a first study of the Calculus; cf. remarks at end.

and $$\int \log x\,dx = x\log x - \int x\frac{dx}{x} = x(\log x - 1) + C.$$

Example 3. To find $\int x\cos^{-1}x\,dx.$

Let $$u = \cos^{-1}x \qquad dv = x\,dx;$$

then $$du = \frac{-dx}{\sqrt{1-x^2}}, \qquad v = \frac{x^2}{2},$$

and $$\int x\cos^{-1}x\,dx = \tfrac{1}{2}x^2\cos^{-1}x + \tfrac{1}{2}\int\frac{x^2\,dx}{\sqrt{1-x^2}}.$$

This last integral can be evaluated by a simple device Write the integrand in the form:

$$\frac{x^2}{\sqrt{1-x^2}} = \frac{1-1+x^2}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2}.$$

It follows, then, that

$$\int\frac{x^2\,dx}{\sqrt{1-x^2}} = \int\frac{dx}{\sqrt{1-x^2}} - \int\sqrt{1-x^2}\,dx$$
$$= \sin^{-1}x - \tfrac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1}x\right]$$
$$= \tfrac{1}{2}\sin^{-1}x - \tfrac{1}{2}x\sqrt{1-x^2}.$$

Since the function $\cos^{-1}x$ already appears in the result, it would be well to replace $\sin^{-1}x$ by its value, $\frac{1}{2}\pi - \cos^{-1}x$. Hence

$$\int\frac{x^2\,dx}{\sqrt{1-x^2}} = -\tfrac{1}{2}\cos^{-1}x - \tfrac{1}{2}x\sqrt{1-x^2} + \tfrac{1}{4}\pi.$$

Substituting this value above, we find:

$$\int x\cos^{-1}x\,dx = \tfrac{1}{4}(2x^2-1)\cos^{-1}x - \tfrac{1}{4}x\sqrt{1-x^2} + C.$$

The method of integration by parts is, however, of exceedingly restricted application for the beginner, and he should be warned against attempting to use it. In fact, the chief applications of the method are those actually given in the examples and exercises of this paragraph. The student's first thought,

in trying to evaluate a new integral, should be that of simplifying it, if possible, by some obvious reduction (in plain English, he should use *gumption*) and then applying, if necessary, the method of substitution. If the integral can be evaluated at all, the chances are ninety-nine to a hundred that he will thus be able to reduce it to a form he can treat, or else to a form given in the *Tables*, § 8.

EXERCISES

Evaluate the following integrals:

1. $\int xe^{ax}\,dx.$
2. $\int x^2 e^{ax}\,dx.$
3. $\int x^3 e^{ax}\,dx.$
4. $\int x \sin x\,dx.$
5. $\int x \cos ax\,dx.$
6. $\int \sin^{-1} x\,dx.$
7. $\int \tan^{-1} x\,dx.$
8. $\int x \sin^{-1} x\,dx.$
9. $\int x \tan^{-1} x\,dx.$
10. $\int x \log x\,dx.$
11. $\int e^{ax} \sin x\,dx.$
12. $\int e^{ax} \cos x\,dx.$

8. Use of the Tables. The integrals that ordinarily arise in practice and which can be evaluated in terms of the elementary functions can be found in such a table of integrals as Peirce's,* and for this reason it is not necessary for us to go further into the theory of integration at this stage. We have learned how to differentiate all the elementary functions, but not all these functions can be integrated in terms of the elementary functions. Thus the integral:

$$\int \frac{dx}{\sqrt{1+x^4}}, \qquad \text{or} \qquad \int \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \quad (0<k^2<1),$$

leads to a new class of transcendental functions, the Elliptic Integrals, and cannot be evaluated in terms of algebraic functions, sines and cosines, etc.

* B. O. Peirce, *A Short Table of Integrals*, Revised Edition, 1910, Ginn & Co., Boston.

There are, however, large classes of functions that can be integrated,* and the classes that are important in practice have been tabulated. The student is requested to examine with care the classification in the *Tables* above referred to.

Example 1. To find by aid of the *Tables* $\int \frac{x\,dx}{(1-x)^3}$.

The integrand is a rational function of x, and so we look under "II. Rational Algebraic Functions," p. 5. There we find "A. — Expressions Involving $a + bx$." Formula 31 gives us the integral we want:

$$\int \frac{x\,dx}{(1-x)^3} = -\frac{1}{1-x} + \frac{1}{2(1-x)^2} + C.$$

Example 2. To find $\int \frac{dx}{1+x+x^2}$.

Here the integrand involves rationally an expression of the form $X = a + bx + cx^2$, and so we look under C, p. 10. Two formulas, 67 and 68, give this integral. But since $q = 4ac - b^2 = 3$ is positive, the second formula would introduce imaginaries. The first gives:

$$\int \frac{dx}{1+x+x^2} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C.$$

It is to be noted that, in the case of inverse trigonometric functions occurring in the *Tables*, the principal value is to be used.

Example 3. To find $\int \frac{dx}{\sqrt{1+x+x^2}}$.

Here the integrand involves $\sqrt{X}$, and so we look under "III. Irrational Algebraic Functions," and find under D, p. 23, Formulas 160, 161. Since $c = 1 > 0$, we choose No. 160:

* When we say, a function *can be integrated*, we mean, can be integrated *in terms of the elementary functions*. Every continuous function has an integral, for the area under its graph is an integral.

$$\int \frac{dx}{\sqrt{1+x+x^2}} = \log\left(\sqrt{1+x+x^2}+x+\frac{1}{2}\right)+C.$$

Example 4. To find $\int \sqrt{a^2+x^2}\,dx.$

It is true that here, too, the integrand involves $\sqrt{X}$, where X is a quadratic polynomial, a^2+x^2. But the linear term in x is lacking in X, and for this reason the formula is simplified. On p. 20 we find "C.—Expressions involving $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2-x^2}$." Formula 124 yields the desired result:

$$\int \sqrt{a^2+x^2}\,dx = \tfrac{1}{2}[x\sqrt{a^2+x^2}+a^2\log(x+\sqrt{a^2+x^2})]+C.$$

Example 5. To find $\int \sin^6 x\,dx.$

The integrand is a transcendental function. Turning to V, p. 35, and looking down the list we come to No. 263:

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

If we set here $n=6$, we reduce the given integral to an expression involving $\int \sin^4 x\,dx$, and this integral can in turn be reduced by the same formula, written for $n=4$. Thus we get finally:

$$\int \sin^6 x\,dx = -\frac{\sin^5 x\cos x}{6}-\frac{5\sin^3 x\cos x}{24}-\frac{5\sin x\cos x}{16}+\frac{5x}{16}.$$

Example 6. To find $\int \frac{dx}{5-4\cos x}.$

Formula 300 gives:

$$\int \frac{dx}{5-4\cos x} = \frac{2}{3}\tan^{-1}\left[3\tan\frac{x}{2}\right]+C.$$

EXERCISES

Evaluate the following integrals with the aid of the *Tables*.

1. $\int \frac{x\,dx}{(4-5x)^2}$. *Ans.* $\frac{1}{25}\left[\log(4-5x)+\frac{4}{4-5x}\right]+C.$

2. $\int \frac{dx}{x^2(1-x)}$. *Ans.* $-\frac{1}{x}+\log\frac{x}{1-x}+C.$

3. $\int \frac{dx}{(x-2)(x-3)}$.

4. $\int \frac{x\,dx}{x^2-5x+6}$.

5. $\int \frac{dx}{5+3x^2}$. *Ans.* $\frac{1}{\sqrt{15}}\tan^{-1}\left(x\sqrt{\frac{3}{5}}\right)+C.$

6. $\int \frac{dx}{5-3x^2}$.

7. $\int \frac{x\,dx}{1+x+x^2}$.

8. $\int \frac{dx}{x+x^2+x^3}$.

Ans. $\frac{1}{2}\log\frac{x^2}{1+x+x^2}-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C.$

9. $\int \frac{\sqrt{1-x}}{x}\,dx$. *Ans.* $2\sqrt{1-x}+\log\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}+C.$

10. $\int \frac{dx}{x\sqrt{x-1}}$.

11. $\int \frac{dx}{x\sqrt{1+x^2}}$.

12. $\int \frac{\sqrt{x^2-5}}{x}\,dx$.

13. $\int \frac{\sqrt{1+4x^2}}{x}\,dx$.

14. $\int \sqrt{-1+4x-x^2}\,dx$.

Ans. $\left(\frac{1}{2}x-1\right)\sqrt{-1+4x-x^2}+\frac{3}{2}\sin^{-1}\frac{x-2}{\sqrt{3}}+C.$

15. $\int \frac{dx}{(7-9x+2x^2)^{\frac{3}{2}}}$.

16. $\int \frac{dx}{x\sqrt{x^2+px+q}}$.

17. $\int \frac{dx}{(1-x^2)\sqrt{1+x^2}}$.

18. $\int \frac{x\,dx}{(1+x^2)\sqrt{1-x^2}}$.

19. $\int \sqrt{\frac{1+x}{x}}\,dx$.

20. $\int \sqrt{\frac{x}{1-x}}\,dx$.

21. $\int \cos^4\theta\, d\theta.$

22. $\int \sin^2\theta \cos^2\theta\, d\theta.$

23. $\int \frac{d\theta}{\sin^4\theta}.$

24. $\int \frac{d\theta}{\cos^4\theta}.$

25. $\int \frac{d\phi}{5 + 7\cos\phi}.$

26. $\int \frac{d\psi}{5 - 4\sin\psi}.$

27. $\int (1 - x^2)e^x\, dx.$

28. $\int (1 + x)\sin x\, dx.$

29. $\int \frac{4 - 3x}{e^x}\, dx.$

30. $\int e^{-x}\cos x\, dx.$

31. $\int x \sin x\, dx.$

32. $\int x \sin^{-1} x\, dx.$

33. $\int e^{-\mu t}\sin pt\, dt.$

34. $\int e^{-\kappa t}\cos(nt + \epsilon)\, dt.$

9. Length of the Arc of a Curve. The Integral Calculus enables us to compute the length of the arc of a curve. It has been shown in Chap. V, § 9, that the differential of the arc is given, in Cartesian coordinates, by the formula:

$$ds^2 = dx^2 + dy^2 = \left(1 + \frac{dy^2}{dx^2}\right)dx^2$$

Hence, if s is measured in such a sense along the curve that s increases when x increases,

(1) $$ds = \sqrt{1 + \frac{dy^2}{dx^2}}\, dx.$$

From this result we obtain by integration:

(2) $$s = \int \sqrt{1 + \frac{dy^2}{dx^2}}\, dx.$$

If s is so measured that s decreases as x increases, the minus sign should be prefixed to the integral.

Example 1. To find the length of the arc of the parabola:

$$y = x^2.$$

Here, $$dy = 2x\, dx,$$

and the integrand becomes:

$$\sqrt{1+\frac{dy^2}{dx^2}}=\sqrt{1+4x^2}.$$

Hence the length of the arc is given by the formula:

$$s=\int\sqrt{1+4x^2}\,dx.$$

This integral is nearly of the form of that given by Formula 124 of the *Tables*; all that is needed being to take the factor 4 outside the radical. Thus

$$(3)\qquad s=2\int\sqrt{\tfrac{1}{4}+x^2}\,dx=x\sqrt{\tfrac{1}{4}+x^2}+\tfrac{1}{4}\log(x+\sqrt{\tfrac{1}{4}+x^2})+C.$$

We have not yet said from what point of the curve the arc shall be measured. On this decision depends the value of C. It would be natural to measure s from the vertex of the parabola, which is at the origin. Then $s=0$ when $x=0$. Substituting these values in (3), we get:

$$0=\tfrac{1}{4}\log\tfrac{1}{2}+C,\qquad\text{or}\qquad C=\tfrac{1}{4}\log 2.$$

Thus (3) becomes

$$(4)\qquad s=\tfrac{1}{2}x\sqrt{1+4x^2}+\tfrac{1}{4}\log(2x+\sqrt{1+4x^2}).$$

This formula gives the length of the arc of the parabola, measured from the vertex to *any* point, (x, y), of the curve. In particular, let us find the length to the point (1, 1):

$$s|_{x=1}=\tfrac{1}{2}\sqrt{5}+\tfrac{1}{4}\log(2+\sqrt{5})=1.11803+\tfrac{1}{4}\log 4.23607.$$

Here, log 4.23607 means the *natural*, NOT the *denary* logarithm. Its value can be found from Peirce's *Tables*, p. 136, and it is seen to be:

$$\log 4.23607=1.44364.$$

Hence $$s|_{x=1}=1.47894.$$

As a check on this result we note that the length of the chord is $\sqrt{2}=1.41$, — a minor approximation. A major approximation is given by the length of the broken line formed

by the tangents at the extremities of the arc. Its length is *

$$\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} = 1.62.$$

Hence the desired length must lie between these limits, and this is seen actually to be the case. If the above logarithm had been taken to the base 10, a result would have been obtained which this check would show to be wrong.

EXERCISES

Find the length of arc of each of the following curves:

1. The parabola $\qquad 4y = 3x^2,$

from the vertex to the point (2, 3).

2. The parabola $\qquad x^2 = 2my,$

from the vertex to the point (x, y).

3. The curve $\qquad 27y^2 = x^3,$

from the origin to the point whose abscissa is 15. *Ans.* 19.

4. The curve $\qquad y = \log x,$

from the point (1, 0) to the point whose abscissa is x.

$$\textit{Ans.}\quad s = \sqrt{1+x^2} - \log\frac{1+\sqrt{1+x^2}}{x} - .5328.$$

5. The curve $\qquad y = \log\cos x,$

from the origin to the point $(\pi/4, 0)$. *Ans.* 0.8814.

6. The curve $\qquad y = \log\sin x,$

from $x = \pi/4$ to $x = \pi/2$.

7. The curve $\qquad y = \log(\sin x + \cos x),$

from the origin.

8. The parabola $\qquad y^2 = x,$

from the vertex.

$$\textit{Ans.}\quad s = \tfrac{1}{2}\sqrt{x+4x^2} + \tfrac{1}{8}\log(4\sqrt{x+4x^2} + 8x + 1).$$

* The student should draw the requisite figure and verify the assertion of the text by actual computation

9. The catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$,
from the origin. *Ans.* $s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$.

10. The catenary of question 9, from the origin to the point whose abscissa is a. *Ans.* $1.1752\,a$.

10. Areas in Polar Coordinates. The area bounded by a curve whose equation is given in polar coordinates can be found by a method similar to that of § 1. Let the equation of the curve be

$$(1) \qquad r = f(\theta).$$

Let A denote the variable area bounded by a fixed radius vector, $\theta = \alpha$, the curve (1), and a variable radius vector, OP. Let $\theta = \theta_0$ correspond to a particular position of OP, and give to θ an increment, $\Delta\theta$, marking at the same time the new position, OP', of the radius vector corresponding to $\theta_1 = \theta_0 + \Delta\theta$. Then the area will have received an increment, ΔA, represented by the figure POP'.

We can approximate to this area by means of two circular sectors, one of radius r_0, the other of radius $r_1 = r_0 + \Delta r$. Since the area of a circular sector of radius R and angle ϕ (measured in radians) is $\frac{1}{2}R^2\phi$, we have:

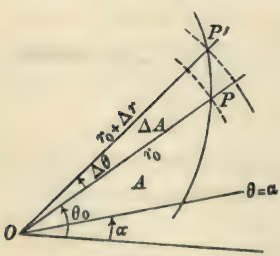


FIG. 77

$$\tfrac{1}{2} r_0^2 \Delta\theta < \Delta A < \tfrac{1}{2} r_1^2 \Delta\theta.$$

Divide through by $\Delta\theta$:

$$\tfrac{1}{2} r_0^2 < \frac{\Delta A}{\Delta\theta} < \tfrac{1}{2}(r_0 + \Delta r)^2.$$

Now allow $\Delta\theta$ to approach 0 as its limit. The ratio $\Delta A/\Delta\theta$ always lies between the fixed quantity $\frac{1}{2} r_0^2$ and the variable $\frac{1}{2}(r_0 + \Delta r)^2$. The latter is approaching the former as its limit, and hence

$$\lim_{\Delta\theta \doteq 0} \frac{\Delta A}{\Delta\theta} = \frac{1}{2} r_0^2.$$

But the limit on the left-hand side is $[D_\theta A]_{\theta=\theta_0}$. Hence, dropping the subscript, we have:

$$(2) \qquad D_\theta A = \tfrac{1}{2} r^2 \qquad \text{or} \qquad dA = \tfrac{1}{2} r^2\, d\theta.$$

If r decreases as θ increases, the inequality signs will be reversed. The reasoning is the same, however, and the result identical.

This formula is analogous to (2), § 1. From it we obtain by integration

$$(3) \qquad A = \tfrac{1}{2} \int r^2\, d\theta.$$

Example. To find the area bounded by the equiangular spiral*

$$r = e^\theta$$

and the radii vectores $\theta = 0$, $\theta = \pi/2$.

Here, $$A = \tfrac{1}{2} \int e^{2\theta}\, d\theta.$$

Hence $$A = \tfrac{1}{4} e^{2\theta} + C.$$

To determine C, observe that $A = 0$ when $\theta = 0$.

Hence $$0 = \tfrac{1}{4} + C, \qquad C = -\tfrac{1}{4},$$

and $$A = \tfrac{1}{4}(e^{2\theta} - 1).$$

This formula gives the area cut off by a *variable* ordinate. To obtain the required area, set $\theta = \pi/2$:

$$A = \tfrac{1}{4}(e^\pi - 1) = 5.53.$$

The student would do well to verify this result by cutting the area out of a piece of cardboard or tin and weighing it.

* The student should make an accurate graph of the curve by means of the tables for e^x, using 1 cm. as the unit. An approximate graph can be made geometrically as follows. Draw rays from the pole, the angle between two consecutive rays being small, — say, 5°. From the point $\theta = 0$, $r = 1$ draw a straight line making an angle of 45° with the prime vector. At the point P_1, where this line cuts the next ray, draw a line making an angle of 45° with OP_1 produced. Repeat.

EXERCISES

1. Find the area of one lobe of the lemniscate

$$r^2 = a^2 \cos 2\theta. \qquad \textit{Ans. } \tfrac{1}{2}a^2.$$

2. Compute the total area inclosed by the cardioid

$$r = a(1 - \cos\phi). \qquad \textit{Ans. } \tfrac{3}{2}\pi a^2.$$

3. Determine the area cut out of the first quadrant by the arc of the equiangular spiral

$$r = ae^{\lambda\theta},$$

corresponding to values of θ from 0 to $\pi/2$.

4. The same for the spiral $r = \theta$.

5. Find the total area inclosed by the curve

$$r^2 = 1 - \theta^2.$$

6. Compute the area of one lobe of the curve

$$r = a \cos 3\theta.$$

7. The same for $\qquad r = a \sin 3\theta.$

8. The same for $\qquad r = a \cos n\theta,$

where n is a natural number.

9. Find the area bounded by the curve

$$r = 10^\theta, \qquad 0 \leqq \theta \leqq \frac{\pi}{6},$$

and the rays $\theta = 0$, $\theta = \pi/6$.

11. Arcs in Polar Coordinates. The differential of the arc of a curve, whose equation in polar coordinates is

$$(1) \qquad r = f(\theta),$$

was found in Chap. V, § 9, to be given by the formula

$$ds^2 = dr^2 + r^2\, d\theta^2.$$

Hence, if s increases with θ,

$$ds = \sqrt{r^2 + \frac{dr^2}{d\theta^2}}\, d\theta,$$

and the length of the arc itself can be obtained by integration:

(2) $$s=\int\sqrt{r^2+\frac{dr^2}{d\theta^2}}\,d\theta.$$

Example 1. To find the length of the arc of the parabola

$$r=\frac{m}{1-\cos\theta}.$$

Here, it is desirable to introduce the half-angle. The details are left to the student. The final result is as follows:

$$s=\frac{m}{4}\left\{-\frac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+\log\tan\frac{\theta}{4}\right\}.$$

Example 2. To find the length of the arc of the equiangular spiral

$$r=e^\theta.$$

Here, it is well to take r as the independent variable, writing the equation of the curve in the form

$$\theta=\log r.$$

Next, we write ds as follows:

(3) $$ds=\sqrt{1+r^2\frac{d\theta^2}{dr^2}}\,dr.$$

Thus the integral takes on the form:

(4) $$s=\int\sqrt{1+r^2\frac{d\theta^2}{dr^2}}\,dr.$$

In the present case,

$$\frac{d\theta}{dr}=\frac{1}{r},$$

and the radicand becomes

$$1+r^2\left(\frac{1}{r^2}\right)=2.$$

Hence $$s=\int\sqrt{2}\,dr=r\sqrt{2}+C.$$

If, in particular, the arc be measured from the point $\theta = 0$, $r = 1$, then

$$0 = \sqrt{2} + C,$$

and

$$s = \sqrt{2}(r - 1).$$

The formula gives a negative result when $r < 1$. This is as it should be, for ds/dr is positive, by (3). Hence s and r increase together, and so, when r decreases, s also decreases. When $r < 1$, the value of s, although negative, will still be *numerically* equal to the length of the arc.

As r approaches 0, the spiral coils about the pole, turning round it an unlimited number of times. Nevertheless, the length of the curve does not increase indefinitely, but approaches $\sqrt{2}$ as its limit:

$$\lim_{r \doteq 0} |s| = \lim_{r \doteq 0} \sqrt{2}\,|r - 1| = \sqrt{2}.$$

EXERCISES

Find the length of arc of each of the following curves.

1. The general equiangular spiral, $r = ae^{\lambda\theta}$, from the point $(a, 0)$ to the point (r, θ). What limit does the length approach when the point (r, θ) approaches the pole?

2. The total length of the cardioid:

$$r = a(1 - \cos\theta). \qquad \textit{Ans.}\ 8a.$$

3. The arc of the spiral $r = \theta$, from the pole to the point where it crosses the prime vector for the first time, $\theta = 2\pi$.

Ans. 21.28.

4. The arc of the spiral $r = 1/\theta$, from the point $(1, 1)$. Compute the length to the point $(2, \frac{1}{2})$, and check your answer.

5. The arc of the curve $r = 1 - \theta$, from the pole to the point where the curve crosses the prime vector for the first time.

CHAPTER X

CURVATURE. EVOLUTES

1. Curvature. We speak of a sharp curve on a railroad and thus express a qualitative characteristic of the curve. Let us see if we cannot get a quantitative determination of the degree of sharpness or flatness of curves in general.

If we consider the angle ϕ by which the tangent of a curve has changed direction when a point that traces out the curve has moved from P to P', then this angle will depend, not only on the sharpness of the curve, but also on the distance from P to P'. We can nearly eliminate this latter element when P' is near P by taking the average change of angle per unit of arc, $\phi/\overset{\frown}{PP'}$. This ratio we define as the average curvature:

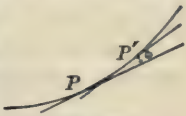


FIG. 78

$$\frac{\phi}{\overset{\frown}{PP'}} = \text{average curvature for arc } PP'.$$

The limit approached by this average curvature is what we understand by the *curvature* at P; it is denoted by κ:

$$\text{(1)} \qquad \kappa = \lim_{P' \doteq P} \frac{\phi}{\overset{\frown}{PP'}} = \text{actual curvature at } P.$$

Thus for a circle of radius a,

$$\overset{\frown}{PP'} = a\phi, \qquad \frac{\phi}{\overset{\frown}{PP'}} = \frac{1}{a}, \qquad \lim_{P' \doteq P} \frac{\phi}{\overset{\frown}{PP'}} = \frac{1}{a} = \kappa,$$

and the average curvature does not change with P'. The curvature of a circle is the same at all points and is equal to the reciprocal of the radius. Thus our numerical definition of curvature, when applied to a circle, jibes well with what we should expect to find, for the larger the radius, the flatter the circle. Furthermore, the curvature of a straight line is 0.

To evaluate the limit (1) for any curve, $y=f(x)$, we observe that, if we write

$$\widetilde{PP'}=\Delta s, \qquad \phi=\Delta\tau,$$

then
$$\kappa=\lim_{\Delta s\doteq 0}\frac{\Delta\tau}{\Delta s}=D_s\tau,$$

where τ denotes as usual the angle which the tangent of the curve makes with the axis of x. More precisely, it is the numerical value of $D_s\tau$ which we want, for κ is an essentially positive quantity (or 0). Hence

(2)
$$\kappa=\pm\frac{d\tau}{ds}, \quad \text{or better:} \quad \kappa=\left|\frac{d\tau}{ds}\right|.$$

From the foregoing definition we see that the curvature is the rate at which the tangent turns when a point describes the curve with unit velocity; for then, $\dfrac{d\tau}{ds}=\dfrac{d\tau}{dt}$.

To compute $d\tau/ds$ we have

(3)
$$\tan\tau=\frac{dy}{dx} \quad \text{or} \quad \tau=\tan^{-1}\frac{dy}{dx}.$$

It will be convenient to introduce a shorter notation for deriva tives and we shall adopt Lagrange's, which employs accents

$$y=f(x),$$

$$\frac{dy}{dx}=y'=f'(x), \qquad \frac{d^2y}{dx^2}=y''=f''(x), \quad \cdots \quad \frac{d^ny}{dx^n}=y^{(n)}=f^{(n)}(x).$$

It follows, then, that

$$dy'=\frac{dy'}{dx}dx=\frac{d^2y}{dx^2}dx=y''\,dx$$

and
$$ds^2=dx^2+dy^2, \qquad ds=\pm\sqrt{1+y'^2}\,dx.$$

Returning to (3) and differentiating we have:

$$\tau=\tan^{-1}y', \qquad d\tau=\frac{dy'}{1+y'^2}=\frac{y''\,dx}{1+y'^2},$$

$$\frac{d\tau}{ds}=\frac{\pm y''}{(1+y'^2)^{\frac{3}{2}}},$$

(4) $$\kappa = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}} = \frac{\left|\dfrac{d^2y}{dx^2}\right|}{\left[1+\dfrac{dy^2}{dx^2}\right]^{\frac{3}{2}}}.$$

The reciprocal of the curvature is called the *radius of curvature* and is usually denoted by ρ: *

(5) $$\rho = \frac{1}{\kappa} = \frac{(1+y'^2)^{\frac{3}{2}}}{|y''|} = \frac{\left[1+\dfrac{dy^2}{dx^2}\right]^{\frac{3}{2}}}{\left|\dfrac{d^2y}{dx^2}\right|}.$$

Analytically it would be simpler not to trouble about the sign and to define:

$$\kappa = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\dfrac{dy^2}{dx^2}\right]^{\frac{3}{2}}}, \qquad \rho = \frac{\left[1+\dfrac{dy^2}{dx^2}\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}.$$

But this analytic simplification is gained only at the expense of geometry, for now κ and ρ are properties no longer of the curve as such; they bear both on the curve and on the choice of the coordinate axes. Thus if the sense of the axis of y is reversed, κ and ρ change their signs; but the curve is the same as before. We prefer, therefore, to retain the definitions of the text, which express the curvature and the radius of curvature as *intrinsic* properties of the curve.

The radius of curvature of a circle is its radius. The curvature of a curve at a point of inflection is in general 0; for $y''=0$ at such a point if y'' is continuous there.

* The student can always recall which of these two ratios is the curvature, which the radius of curvature, by the check of *dimensions*. If we regard x and y each as of the first degree in length, then $y' = dy/dx$ is of the 0-th and $y'' = dy'/dx$ of the $-$1st degree. Hence the bracket is of the 0-th degree and $|y''|$ of the $-$1st, and the ratio must therefore be written so as to yield ρ of the 1st, κ of the $-$1st degree in length.

Example. To find the curvature of the parabola

$$y^2 = 2mx.$$

Here $$2y\,dy = 2m\,dx, \qquad y' = \frac{m}{y};$$

$$dy' = -\frac{m}{y^2}dy, \qquad y'' = -\frac{m^2}{y^3};$$

$$\kappa = \frac{m^2|y|^{-3}}{\left[1+\frac{m^2}{y^2}\right]^{\frac{3}{2}}} = \frac{m^2}{(m^2+y^2)^{\frac{3}{2}}}, \qquad \rho = \frac{(m^2+y^2)^{\frac{3}{2}}}{m^2}.$$

EXERCISES

Find the curvature of each of the following curves.

1. $y = x^2$. *Ans.* $\kappa = \dfrac{2}{(1+4x^2)^{\frac{3}{2}}}$.

2. $y = x^3$, at the origin.

3. $y = \log\csc x$. *Ans.* $\kappa = \sin x$.

4. The ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Ans.* $\kappa = \dfrac{a^4b^4}{(b^4x^2+a^4y^2)^{\frac{3}{2}}}$.

5. The hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. *Ans.* $\kappa = \dfrac{a^4b^4}{(b^4x^2+a^4y^2)^{\frac{3}{2}}}$.

6. The equilateral hyperbola: $xy = \dfrac{a^2}{2}$.

Ans. $\kappa = \dfrac{a^2}{(x^2+y^2)^{\frac{3}{2}}}$.

7. Show that the radius of curvature of the curve $y = x^{\frac{2}{3}}$ approaches 0 as its limit when the point P approaches the cusp, (0, 0).

8. Find the radius of curvature of the curve

$$54y = 10x^5 - 19x^4 + 11x^3 + x^2 - 72x$$

at the origin. *Ans.* $\rho = 125$.

9. Find the radius of curvature of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

at the vertex. *Ans.* $\rho = a$.

10. At what points of the curve $y = x^3$ is the curvature greatest?

11. Find the locus of the points in which the curvature of the curves of Fig. 52, p. 160, $y = x^n$, $x > 0$, $n > 0$, is greatest.

$$\textit{Ans.} \quad x = \left[\frac{n-2}{(2n-1)n^2}\right]^{\frac{1}{2n-2}}, \quad y = \left[\frac{n-2}{(2n-1)n^2}\right]^{\frac{n}{2n-2}}.$$

2. The Osculating Circle. At an arbitrary point P of a curve let the normal be drawn toward the concave side of the curve and let a distance be laid off on this normal equal to the radius of curvature, ρ. The point Q thus obtained is called the *centre of curvature*. The circle constructed with Q as centre and with radius ρ stands in an important relation to the curve. It is called the *osculating circle* and has the property that it represents the curve more accurately near P than any other circle does. Consider the family of circles drawn tangent to the curve at P with their centres on the concave side. Those whose radii are very short are curved too sharply, — more sharply than the given curve. Now let the circles grow. If we pass to the other extreme of circles with very large radii, these will be too flat. Evidently, then, certain intermediate circles come nearer to the shape of the curve at P than these extreme ones do. It is not difficult to find a criterion by means of which one circle is characterized as better than all the others. Draw the tangent at P and drop a perpendicular from P' on it meeting it in M and cutting an arbitrary one of the circles in R. Then, as is shown by means of Taylor's Theorem with the Remainder, MP' will in general be an infinitesimal of the second order referred to the arc PP' as principal infinitesimal, and $P'R$ will also be of the second order for a circle taken at random. We can, however, in general find *one* circle for which $P'R$ will be an infinitesimal of the *third* order, and it turns out that this circle is precisely the osculating circle.

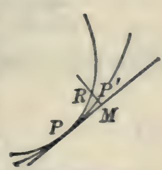


FIG. 79

The osculating circle crosses the curve in general at the point of tangency; but there may be certain exceptional points at which this is not the case. Near such a point $P'R$ is an infinitesimal of even higher order than the third; in general, it is of the fourth.

EXERCISE

Construct carefully the parabola $y = x^2$ for values of x: $-\frac{3}{4} \leqq x \leqq \frac{3}{4}$, taking 10 cm. as the unit. Draw the osculating circle at the point $x = \frac{1}{2}$, $y = \frac{1}{4}$, and also at the vertex. Ink in the parabola in a fine black line, the first osculating circle in red, and the second in a different colored ink or in pencil. State accurately all that the figures appear to show.

3. The Evolute. When a point P traces out a curve, the centre of curvature, Q, traces out a second curve. This latter curve — the locus of Q — is called the *evolute* of the given curve. We proceed to deduce its equation and to discuss its properties.

The point Q can be found analytically by writing down the equation of the normal at P and determining the intersection of this line with a circle of radius ρ, having its centre at P. The equation of the normal is

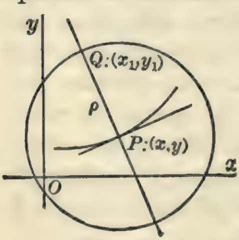


FIG. 80

$$(6) \qquad X - x + y'(Y - y) = 0,$$

where (X, Y) are the running coordinates, *i.e.* the coordinates of a variable point on the normal, and (x, y) the coordinates of P, — the latter being held fast during the following investigation. The equation of the circle is

$$(7) \qquad (X - x)^2 + (Y - y)^2 = \rho^2 = \frac{(1 + y'^2)^3}{y''^2}.$$

To find where (6) and (7) intersect, eliminate X:

$$(1 + y'^2)(Y - y)^2 = \frac{(1 + y'^2)^3}{y''^2}, \qquad Y - y = \pm \frac{1 + y'^2}{y''}.$$

Which sign must we take? Notice that when the curve is concave upward, as in the figure, Y is greater than y, or

$$Y - y > 0, \qquad \text{and} \qquad y'' = \frac{d^2y}{dx^2} > 0.$$

Hence in this case we must use the upper sign:

$$Y - y = \frac{1 + y'^2}{y''}. \tag{8}$$

On the other hand, when the curve is concave downward,

$$Y - y < 0 \qquad \text{and} \qquad y'' = \frac{d^2y}{dx^2} < 0,$$

and again we have the upper sign. Hence (8) is always true and

$$Y = y + \frac{1 + y'^2}{y''}.$$

From (6) and (8) we get:

$$X = x - \frac{y'(1 + y'^2)}{y''}.$$

The values of X and Y thus found are the coordinates (x_1, y_1) of the point Q, and so we have:

$$x_1 = x - \frac{\dfrac{dy}{dx}\left(1 + \dfrac{dy^2}{dx^2}\right)}{\dfrac{d^2y}{dx^2}}, \qquad y_1 = y + \frac{1 + \dfrac{dy^2}{dx^2}}{\dfrac{d^2y}{dx^2}}. \tag{9}$$

These formulas involve no radicals.

Formulas (9) can also be obtained by triangulation and the use of the values deduced for $\sin\tau$ and $\cos\tau$ in Chap. V, § 9.

If we eliminate x and y between the two equations (9) and the equation $y = f(x)$ of the given curve, we shall obtain the equation of the evolute in the form

$$F(x_1, y_1) = 0.$$

But it is not necessary to eliminate. We can plot as many points on the evolute as we like by substituting in (9) the values of x, y, y', and y'' corresponding to successive points on the given curve.

Example 1. To find the evolute of the parabola

$$y^2 = 2mx. \tag{10}$$

Here, $\dfrac{dy}{dx} = \dfrac{m}{y}$, $\quad 1 + \dfrac{dy^2}{dx^2} = \dfrac{m^2 + y^2}{y^2}$, $\quad \dfrac{d^2y}{dx^2} = -\dfrac{m^2}{y^3}$.

Hence $$x_1 = x - \frac{m(m^2+y^2)}{y^3} \Big/ -\frac{m^2}{y^3} = x + \frac{m^2+y^2}{m},$$

$$y_1 = y + \frac{m^2+y^2}{y^3} \Big/ -\frac{m^2}{y^2} = -\frac{y^3}{m^2},$$

and it remains to eliminate x and y between these equations and (10). Eliminating x we have:

$$x_1 = \frac{y^2}{2m} + \frac{m^2+y^2}{m} = m + \frac{3y^2}{2m}$$

or

$$\frac{2m}{3}(x_1 - m) = y^2.$$

From the second equation:

$$-m^2 y_1 = y^3.$$

To eliminate y between these last two equations, square each side of the last and cube each side of the preceding one. Thus we get:

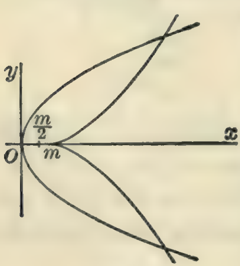


FIG. 81

$$m^4 y_1^2 = \frac{8m^3}{27}(x_1 - m)^3.$$

Dropping the subscripts we have as the equation of the evolute of the parabola:

$$y^2 = \frac{8}{27m}(x - m)^3. \tag{11}$$

This is a so-called *semi-cubical parabola.*

Example 2. To find the evolute of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{12}$$

We obtain without difficulty the equations:

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \qquad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3},$$

$$x_1 = x - \frac{x(b^4x^2 + a^4y^2)}{a^4b^2}, \qquad y_1 = y - \frac{y(b^4x^2 + a^4y^2)}{a^2b^4}.$$

To eliminate x and y between these equations and (12) requires a little ingenuity. From (12) we have

$$b^2x^2 + a^2y^2 = a^2b^2, \qquad a^4y^2 = a^2b^2(a^2 - x^2),$$

$$b^4x^2 + a^4y^2 = b^2(a^4 - a^2x^2 + b^2x^2).$$

Hence $$x_1 = x - \frac{b^2x(a^4 - a^2x^2 + b^2x^2)}{a^4b^2} = \frac{a^2 - b^2}{a^4}x^3.$$

In a similar manner we get:

$$y_1 = y - \frac{a^2y(b^4 - b^2y^2 + a^2y^2)}{a^2b^4} = -\frac{a^2 - b^2}{b^4}y^3.$$

We can solve these equations respectively for x^2 and y^2 and substitute the values thus obtained in (12):

$$\left(\frac{ax_1}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{by_1}{a^2 - b^2}\right)^{\frac{2}{3}} = 1.$$

Dropping the accents we have as the final equation of the evolute of the ellipse:

(13) $\quad (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$

FIG. 82

EXERCISES

Find the equation of the evolute of each of the following curves.

1. The hyperbola: $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Ans. $\quad (ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$

2. The hyperbola: $\quad 2xy = a^2.$

$$\textit{Ans.} \quad (x+y)^{\frac{2}{3}} - (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

3. The catenary: $\quad y = \frac{1}{2}(e^x + e^{-x}).$

$$\textit{Ans.} \quad x_1 = x - \tfrac{1}{4}(e^{2x} - e^{-2x}), \quad y_1 = 2y;$$

$$x = \log\left[\frac{y}{2} \pm \sqrt{\frac{y^2}{4} - 1}\right] \mp \frac{y}{2}\sqrt{\frac{y^2}{4} - 1}.$$

4. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$ $\qquad$ *Ans.* $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$

5. $x = a(\cos\theta + \theta\sin\theta),$
$y = a(\sin\theta - \theta\cos\theta).$ $\qquad$ *Ans.* $x^2 + y^2 = a^2.$

6. $x = a\cos^3\theta, \quad y = a\sin^3\theta.$ $\quad$ *Ans.* $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$

4. Properties of the Evolute. The property of the evolute to which the curve owes its name is the following. Suppose a material cylinder to be constructed on the concave side of the evolute and a string to be wound on the cylinder, Fig. 83. Let a pencil be fastened to the end of the string, the point being placed at a point P_0 of the given curve and the string drawn taut and fastened at a point A of the evolute so that it cannot slip. If now the pencil is moved along the paper so that the string unwinds from the evolute or winds up, the pencil will describe the given curve.

To prove this, let P be an arbitrary point of the given curve, Q the corresponding point of the evolute, and P' the position of the pencil when the string leaves the evolute at Q. We wish to prove that P' coincides with P. To do this it is sufficient to show (*a*) that QP is tangent to the evolute, so that P' lies on QP; and (*b*) that $QP' = QP = \rho$.

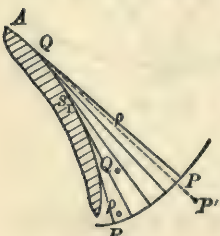


FIG. 83

ad (*a*) Writing equations (9) in the form:

$$x_1 = x - \frac{y'(1+y'^2)}{y''}, \qquad y_1 = y + \frac{1+y'^2}{y''}$$

and differentiating with respect to x, we have:*

$$\frac{dx_1}{dx} = x_1' = \frac{y'(1+y'^2)y'''}{y''^2} - 3y'^2 = \frac{y'(1+y'^2)y''' - 3y'^2y''^2}{y''^2},$$

$$\frac{dy_1}{dx} = y_1' = 3y' - (1+y'^2)\frac{y'''}{y''^2} = \frac{3y'y''^2 - (1+y'^2)y'''}{y''^2},$$

$$\therefore\ \frac{dy_1}{dx_1} = -\frac{1}{y'} = -\frac{1}{\dfrac{dy}{dx}},$$

and thus the slope of the evolute at Q is seen to be the negative reciprocal of the slope of the given curve at P. Hence QP is tangent to the evolute, q. e. d.

ad (*b*) If we denote by s_1 the length of the arc Q_0Q of the evolute, then $QP' = s_1 + \rho_0$, and we wish to show that this quantity is equal to ρ:

$$s_1 + \rho_0 = \rho.$$

It is evidently sufficient to show that

$$\frac{ds_1}{dx} = \frac{d\rho}{dx}.$$

Now
$$ds_1^2 = dx_1^2 + dy_1^2,$$

$$\frac{ds_1^2}{dx^2} = x_1'^2 + y_1'^2 = y_1'^2\left(1 + \frac{x_1'^2}{y_1'^2}\right) = y_1'^2(1+y'^2).$$

And again:

$$\frac{d\rho}{dx} = \pm\frac{3y'y''^2 - (1+y'^2)y'''}{y''^2}\sqrt{1+y'^2} = \pm\, y_1'\sqrt{1+y'^2}.$$

Hence
$$\frac{ds_1}{dx} = \pm\frac{d\rho}{dx},$$

and since we have taken s_1 so that it increases when ρ increases, the upper sign holds:

$$ds_1 = d\rho, \qquad s_1 = \rho + C.$$

* The student may find it more convenient in working out these differentiations to retain the form (9). Lagrange's form is more compact.

At $\quad Q_0,\quad s_1 = 0$ and $\rho = \rho_0,\quad$ hence $0 = \rho_0 + C,$
and

$$\rho = s_1 + \rho_0, \qquad \text{q. e. d.}$$

We have shown incidentally that the normals to the given curve are tangent to the evolute. Thus it appears that the evolute is the envelope of the normals of the given curve. This property can be used as the definition of the evolute and its equation is then readily deduced by the method of envelopes; cf. the chapter on Envelopes in the second volume.

EXERCISES

1. If the equation of the curve is given in polar coordinates, $r = f(\theta)$, then (see Fig. 45)

$$\Delta\tau = \Delta\psi + \Delta\theta$$

and hence
$$\frac{d\tau}{ds} = \frac{d\psi}{ds} + \frac{d\theta}{ds}.$$

Remembering that

$$\tan\psi = r\frac{d\theta}{dr} = \frac{r}{r'},$$

where $r' = dr/d\theta$, obtain the formula,

$$\rho = \pm\frac{\left[r^2 + \frac{dr^2}{d\theta^2}\right]^{\frac{3}{2}}}{r^2 - r\frac{d^2 r}{d\theta^2} + 2\frac{dr^2}{d\theta^2}}. \tag{14}$$

Find the radius of curvature of each of the following curves at any point.

2. The spiral of Archimedes $r = a\theta$. *Ans.* $\rho = \dfrac{(r^2 + a^2)^{\frac{3}{2}}}{r^2 + 2a^2}.$

3. The cardioid $r = 2a(1 - \cos\phi)$. *Ans.* $\rho = \frac{4}{3}\sqrt{ar}.$

4. The lemniscate $r^2 = a^2\cos 2\theta$. *Ans.* $\rho = \dfrac{a^2}{3r}.$

5. The equilateral hyperbola $r^2\cos 2\theta = a^2$. *Ans.* $\rho = \dfrac{r^3}{a^2}.$

6. The equiangular spiral $r = ae^{\lambda\theta}$.

7. The trisectrix $r = 2a\cos\theta - a$.

$$Ans.\quad \rho = \frac{a(5 - 4\cos\theta)^{\frac{3}{2}}}{9 - 6\cos\theta}.$$

8. If the equation of a curve be written in the form: $x = \phi(s)$, $y = \psi(s)$, show that

$$\pm\kappa = \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}.$$

CHAPTER XI

THE CYCLOID

1. The Equations of the Cycloid. The cycloid is the path traced out by a point in the rim of a wheel as it rolls, *i.e.* by a point in the circumference of a circle which rolls without slipping on a straight line, always remaining in the same plane. Let the given line be taken as the axis of x and let θ be the angle through which the circle has turned since the point P was last in contact with the line at O. The coordinates of P,

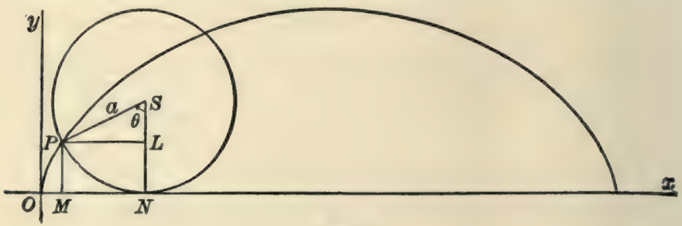


FIG. 84

$x = OM$ and $y = MP$, can be expressed as follows in terms of θ. We notice that the arc NP of the circle and the segment ON of the line are of equal length, $a\theta$, since the circle rolls without slipping. Hence

$$OM = ON - MN = a\theta - a\sin\theta.$$

Also, $$MP = NS - LS = a - a\cos\theta;$$

and so we have:

$$(1) \qquad \begin{cases} x = a(\theta - \sin\theta), \\ y = a(1 - \cos\theta), \end{cases}$$

as the equations of the cycloid.

It is possible to eliminate θ between these equations and thus obtain a single equation between x and y. But the func-

tions thus introduced are less simple than those of equations (1) and it is more convenient to discuss the properties of the curve directly by means of these equations.

EXERCISES

1. Cut out a circular disc, 1 inch in diameter, from a piece of cardboard of medium thickness; mark a point on the rim of the disc, and then, holding a straight-edge firmly on the paper, cause the disc to roll without slipping on the former. Mark on the paper a good number of the positions of the point, and then draw a clean, firm curve through them. Also draw the line of the straight-edge.

From the cycloid thus constructed prepare a templet; cf. Analytic Geometry, p. 89, Ex. 2.

2. The equations of the cycloid referred to parallel axes with the new origin at the vertex, *i.e.* the highest point, are:

$$(2) \qquad \begin{cases} x = a\theta + a\sin\theta, \\ y = -a + a\cos\theta, \end{cases}$$

the angle θ now being the angle through which the circle has turned since the point P was at the vertex. Obtain these equations geometrically, drawing first the requisite figure, and verify the result analytically by transforming the equations (1):

$$x = x' + \pi a, \qquad y = y' + 2a, \qquad \theta = \theta' + \pi.$$

3. Show that the equations of an inverted cycloid referred to the vertex as origin can be written in the form:

$$(3) \qquad \begin{cases} x = a\theta + a\sin\theta, \\ y = a - a\cos\theta. \end{cases}$$

Draw the figure and interpret θ geometrically.

2. **Properties of the Cycloid.** The slope of the curve is

$$\frac{dy}{dx} = \frac{a\sin\theta\, d\theta}{a d\theta - a\cos\theta\, d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \cot\tfrac{1}{2}\theta,$$

or

$$\tan\tau = \cot\tfrac{1}{2}\theta$$

From this result we infer that the tangent at P is perpendicular to the chord PN, Fig. 85. For the latter makes an angle of $\frac{1}{2}\theta$ with the negative axis of x and hence its slope is $-\tan\frac{1}{2}\theta$, *i.e.* the negative reciprocal of the slope of the tangent. Thus we see that the normal at P goes through the lowest point of the generating circle and hence the tangent at P goes through the highest point.

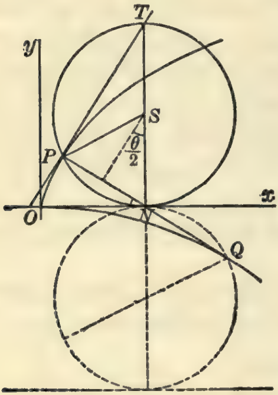


FIG. 85

The equation of the tangent at the point (x_0, y_0), $\theta = \theta_0$, is

$$(4)\qquad y - y_0 = \cot\tfrac{1}{2}\theta_0(x - x_0),$$

and of the normal:

$$(5)\quad x - x_0 + \cot\tfrac{1}{2}\theta_0(y - y_0) = 0.$$

The Evolute. We have seen that $\dfrac{dy}{dx} = \cot\frac{1}{2}\theta$. Hence $1 + \dfrac{dy^2}{dx^2} = \csc^2\frac{1}{2}\theta$

and
$$\frac{d^2y}{dx^2} = \frac{d\dfrac{dy}{dx}}{dx} = \frac{-\frac{1}{2}\csc^2\frac{1}{2}\theta\,d\theta}{a\,d\theta - a\cos\theta\,d\theta} = -\frac{1}{4a\sin^4\frac{1}{2}\theta},$$

$$(6)\quad \rho = \frac{4a\sin^4\frac{1}{2}\theta}{\sin^3\frac{1}{2}\theta} = 4a\sin\tfrac{1}{2}\theta.$$


FIG. 86

It is now easy to construct the centre of curvature and thus find the evolute. We have merely to lay off on the normal PN a distance $PQ = 4a\sin\frac{1}{2}\theta$, *i.e.* double the distance PN. The locus of the point Q is thus seen to be an equal cycloid having its vertex at the origin O. We leave the proof, which is simple, to the student, referring him to Fig. 85.

The Arc. We have:

$$ds^2 = dx^2 + dy^2 = a^2[(1-\cos\theta)^2 + \sin^2\theta]\,d\theta^2 = 4a^2\sin^2\tfrac{1}{2}\theta\,d\theta^2,$$

$$s = 2a\int \sin\tfrac{1}{2}\theta\, d\theta = 4a\int \sin\tfrac{1}{2}\theta\, d(\tfrac{1}{2}\theta) = -4a\cos\tfrac{1}{2}\theta + C.$$

If we measure the arc from the origin,

$$0 = -4a + C, \qquad C = 4a,$$

(7) $$\therefore\ s = 4a(1 - \cos\tfrac{1}{2}\theta) = 8a\sin^2\tfrac{1}{4}\theta.$$

The total length of one arch of the cycloid is, therefore, $8a$.

Area of an Arch. This area was first determined experimentally by Galileo, who cut out an arch and weighed it. We can find the area under the curve by integration:

$$A = \int y\,dx = \int [a - a\cos\theta][a\,d\theta - a\cos\theta\,d\theta]$$

$$= a^2\int (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= a^2[\theta - 2\sin\theta + \tfrac{1}{2}(\theta + \sin\theta\cos\theta)] + C,$$

$$0 = 0 + C,$$

(8) $$\therefore\ A = a^2(\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{2}\sin\theta\cos\theta).$$

The area of the complete arch is, therefore, $3\pi a^2$, or three times that of the generating circle.

3. The Epicycloid and the Hypocycloid. When a circle rolls without slipping on a second circle that is fixed, always remaining in its plane and tangent to it externally, a point P in the circumference of the moving circle traces out an *epicycloid.* From Fig. 87 it is clear that

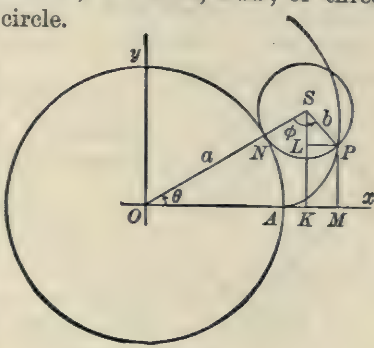


FIG. 87

$$x = OK + KM = (a+b)\cos\theta + b\sin\left[\phi - \left(\frac{\pi}{2} - \theta\right)\right],$$

$$y = KS - LS = (a+b)\sin\theta - b\cos\left[\phi - \left(\frac{\pi}{2} - \theta\right)\right].$$

Furthermore, the arc AN and the arc NP are equal; so $a\theta = b\phi$. Hence we have as the equations of the epicycloid:

$$(9)\qquad \begin{cases} x = (a+b)\cos\theta - b\cos\dfrac{a+b}{b}\theta, \\ y = (a+b)\sin\theta - b\sin\dfrac{a+b}{b}\theta. \end{cases}$$

If the variable circle rolls on the inside of the fixed circle, the path of the point P is a *hypocycloid.* Its equations are obtained in a similar manner and are:

$$(10)\qquad \begin{cases} x = (a-b)\cos\theta + b\cos\dfrac{a-b}{b}\theta, \\ y = (a-b)\sin\theta - b\sin\dfrac{a-b}{b}\theta. \end{cases}$$

The following special cases are of interest.

(1) If $a = 2b$, the hypocycloid reduces to a segment of a straight line, namely, the diameter of the circle, $y = 0$. Thus a journal on the rim of a toothed wheel which meshes internally with another wheel of twice the diameter describes a right line, so that circular motion is thereby converted into rectilinear motion.

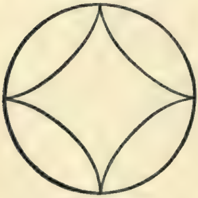


Fig. 88

(2) When $a = 4b$, the equations of the hypocycloid reduce to the following (cf. *Tables*, Formulas 580 and 585):

$$\begin{cases} x = 3b\cos\theta + b\cos 3\theta = a\cos^3\theta, \\ y = 3b\sin\theta - b\sin 3\theta = a\sin^3\theta. \end{cases}$$

Hence $$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

This is the equation of the *four-cusped hypocycloid.*

The cycloids play an important part in Applied Mechanics, in the theory of the shape in which the teeth of gears should be cut.

For a more extensive discussion of the subject of this chapter see Williamson, *Differential Calculus,* Chap. XIX, Roulettes.

EXERCISES

1. Show by means of the equation of the normal of the cycloid, (5), that the normal goes through the lowest point of the generating circle.

2. Obtain the equations of the path of the journal of the driver of a locomotive and plot the curve.

3. Obtain the equations of the path of a point on the outer edge of the flange of a driver.

4. Obtain the equations of the path of the pedal of a bicycle.

5. Obtain the equation of the path of an arbitrary point in the wheels of a sidewheel steamboat.

The curves of Exs. 2–5 are called *trochoids.*

6. Find the velocity, v, of the point that generates a cycloid.

Ans. $v = 2a\omega \sin \frac{1}{2}\theta = 2V \sin \frac{1}{2}\theta$, where ω is the angular velocity of the wheel and V the linear velocity of the hub. At the vertex $v = 2V$, *i.e.* the velocity of the highest point of the wheel is twice that of the hub.

7. Find the area included between an arch of the cycloid and its evolute. *Ans.* $4\pi a^2$.

8. Show that the length of the arc of an inverted cycloid (3), measured from the vertex is

$$s = 4a \sin \tau.$$

9. Obtain the equations of the evolute of the cycloid analytically, by means of equations (9) in Chap. X.

10. At what point is the trochoid

$$x = a\theta - b \sin\theta, \qquad y = a - b\cos\theta, \qquad (b < a)$$

steepest? *Ans.* When $\cos\theta = \frac{b}{a}$.

11. Find the area under one arch of the trochoid of question 10. *Ans.* $2\pi a^2 + \pi b^2$.

12. The epicycloid for which $b = a$ is a cardioid:

$$r = 2a(1 - \cos\phi),$$

the cusp being taken as the pole. Obtain this result from equations (9).

13. Obtain the result in question 12 directly geometrically.

14. Prove by elementary geometry that the hypocycloid for which $b = \frac{1}{2}a$ is a straight line.

15. Show that the equation of the normal of the hypocycloid is:

$$\left(\sin\theta_0 + \sin\frac{a-b}{b}\theta_0\right)(x - x_0) = \left(\cos\theta_0 - \cos\frac{a-b}{b}\theta_0\right)(y - y_0).$$

16. Prove that the normal of the hypocycloid passes through the point of contact of the rolling circle.

17. Work out questions 15 and 16 for the epicycloid.

18. Show that the hypocycloid for which $b = \frac{1}{3}a$ and that for which $b = \frac{2}{3}a$ are the same curve.

19. Show that the length of the four-cusped hypocycloid is three times the diameter of the fixed circle.

20. Find the area of the four-cusped hypocycloid.

Ans. $\dfrac{3\pi a^2}{8}$.

21. Find the area enclosed between one arch of an epicycloid and the fixed circle. *Ans.* $\dfrac{7\pi a^2}{32}$.

22. Obtain the equations of the epitrochoid.

23. Obtain the equations of the hypotrochoid.

24. How many revolutions does the rolling circle make in tracing out a cardioid? a four-cusped hypocycloid? How many revolutions does the moon make in a lunar month?

25. How many cusps does an epicycloid have when a and b are commensurable: $a/b = p/q$? What can you say about this curve when a and b are incommensurable?

CHAPTER XII

DEFINITE INTEGRALS

1. The Area under a Curve by the Earlier Method. In Chapter IX we learned a method for computing the area A under a curve, $y=f(x)$. We found that

$$D_x A = y,$$

and hence
$$A=\int y\,dx+C,$$

where we mean by the first term on the right a particular integral which, once chosen, shall be retained throughout what follows.

In order to determine the constant C of integration, we observed that, as x approaches a as its limit, A approaches 0, and hence C must have such a value as to satisfy the equation

$$0=\left[\int y\,dx\right]_{x=a}+C, \qquad \text{or} \qquad C=-\left[\int y\,dx\right]_{x=a}.$$

Thus the variable area A has the value

$$A=\int y\,dx-\left[\int y\,dx\right]_{x=a}.$$

To find the value of the area A we set out to compute, it remains merely to put $x=b$ in this formula:

$$A=\left[\int y\,dx\right]_{x=b}-\left[\int y\,dx\right]_{x=a}.$$

As a matter of notation the right-hand side of this equation is abbreviated as follows:

$$\left[\int y\,dx\right]_{x=b}-\left[\int y\,dx\right]_{x=a}=\left[\int y\,dx\right]_{x=a}^{x=b}.$$

Hence

(1) $$A=\left[\int y\,dx\right]_{x=a}^{x=b}.$$

We have said here nothing new. We have merely formulated the method set forth in the earlier chapter. Take the example of § 1 of that chapter. Here, $y = x^2$, and we should take most naturally as the particular integral

$$\int y\,dx = \frac{x^3}{3}.$$

Then, $$A = \left[\frac{x^3}{3}\right]_{x=1}^{x=2} = \left[\frac{x^3}{3}\right]_{x=2} - \left[\frac{x^3}{3}\right]_{x=1} = \frac{8}{3} - \frac{1}{3} = 2\tfrac{1}{3}.$$

The notation just explained applies to any function in the brackets, whether it is thought of as arising by integration or not. Thus

$$\Big[2x + 3\Big]_{x=2}^{x=5} = 13 - 7 = 6.$$

An equivalent notation in common use is the following:

$$2x + 3\,\Big|_2^5 = 13 - 7 = 6.$$

It applies to any function:

$$\Big[\phi(x)\Big]_{x=a}^{x=b} = \phi(x)\,\Big|_a^b.$$

Thus $$2x - \cos x\,\Big|_0^{\frac{\pi}{2}} = \pi + 1.$$

EXERCISES

1. Show that if, in the case of the area computed in the above example, the particular integral

$$\int y\,dx = \int x^2\,dx = \frac{x^3}{3} - 1$$

had been chosen, Formula (1) would have yielded the same result. The same for the choice $\frac{1}{3}x^3 + 5$. Generalize.

Compute the value of each of the following expressions.

2. $\Big[\sin x\Big]_0^{\frac{\pi}{2}}$. *Ans.* 1.

3. $\Big[3\log x\Big]_1^e$. *Ans.* 3.

4. $$x^3 + 2x - 8 \Big|_a^b.$$

Ans. $b^3 - a^3 + 2(b - a)$

5. $$e^{-x} \cos x - \sin \frac{x}{2} \Big|_0^{\pi}.$$ *Ans.* $-e^{-\pi} - 2.$

2. A New Expression for the Area under a Curve. Let

$$y = f(x)$$

be a continuous* function of x, and let the area under the curve be divided into strips as indicated in Fig. 89. This is done by dividing the interval (a, b) of the axis of x:

$$a \leqq x \leqq b,$$

into n equal parts by the points $x_0 = a,\ x_1,\ x_2, \cdots,\ x_{n-1},\ x_n = b$ and erecting ordinates at the points of division. Let each strip be replaced by a rectangle whose altitude is the left-hand ordinate of the strip. Then the sum of the areas of these rectangles will be approximately equal to the area A in question, and will approach A as its limit when n is allowed to increase without limit.

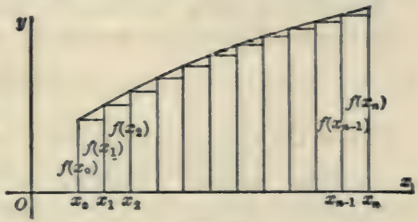


FIG. 89

* Throughout this chapter, whenever a function is required to be *continuous*, it is understood that the further restriction is imposed that the function is monotonic (p. 207) throughout the entire interval, or else that its graph is made up of a finite number of pieces, each of which represents a function which is monotonic or constant in the corresponding subinterval; *e.g.*

$$f(x) = x^2 \quad \text{when} \quad 0 \leqq x \leqq 1; \qquad f(x) = 1 \quad \text{when} \quad 1 < x \leqq 2.$$

It is true that the definition of the definite integral, and the Fundamental Theorem, hold for any continuous function. But a first course in the Calculus is not an appropriate place for discussing all the possibilities presented by continuous functions in their entire generality.

Let us formulate this sum analytically. Denote by Δx the length of the base of a rectangle:

$$\Delta x = \frac{b-a}{n}.$$

Then the area of the first rectangle is

$$f(a)\,\Delta x \qquad \text{or} \qquad f(x_0)\,\Delta x.$$

The area of the second rectangle is $f(x_1)\,\Delta x$, and so on. Hence the sum in question has the value

$$(1) \qquad f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x.$$

We are now ready to allow n to increase without limit. The limit approached by this sum is the area A under the curve:

$$(2) \qquad A = \lim_{n=\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x].$$

This is the new expression for A which we set out to obtain.

Remarks. Instead of taking the left-hand ordinate of the strip as the altitude of the rectangle, we might equally well take the right-hand ordinate. The area of the first rectangle would then be $f(x_1)\,\Delta x$; that of the second, $f(x_2)\,\Delta x$, etc.; and the sum in question would have the value

$$(3) \qquad f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x.$$

The expression for the area A now assumes the form: *

$$(4) \qquad A = \lim_{n=\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x].$$

We could go even further and take the sub-intervals (x_k, x_{k+1}) of unequal lengths, setting

* The student may wonder why we do not approximate to A by trapezoids formed by drawing the chord for each strip, instead of using the flight of stairs, — the inscribed or the circumscribed rectangles. This is precisely what we shall do later (cf. § 22) when the most accurate numerical approximation conveniently attainable is our objective. Here, however, it is the *variable* represented by the formula (1) or (3) or (5) in which we are interested, and whose limit we desire. The above geometric interpretation enables us to study that variable and its limit.

$$\Delta x_k = x_{k+1} - x_k.$$

Moreover, we could take the ordinate corresponding to any point x'_k of the k-th interval as the altitude of the k-th rectangle.

Then
$$x_k \leqq x'_k \leqq x_{k+1},$$

and the area of the k-th rectangle is $f(x'_k)\,\Delta x_k$.

Thus we have the sum

(5) $$f(x'_0)\,\Delta x_0 + f(x'_1)\,\Delta x_1 + \cdots + f(x'_{n-1})\,\Delta x_{n-1}.$$

The limit of this sum when n increases indefinitely, the longest sub-interval approaching 0 as its limit, is the area A:

(6) $$A = \lim_{n=\infty}[f(x'_0)\,\Delta x_0 + f(x'_1)\,\Delta x_1 + \cdots + f(x'_{n-1})\,\Delta x_{n-1}].$$

Example. Let $f(x) = \sin x$,

$$y = \sin x,$$

and let the interval (a, b) be the interval $(0, \pi/2)$:

$$0 \leqq x \leqq 1.57$$

Let $n = 10$; then

$$\Delta x = \frac{b-a}{n} = \frac{1.57}{10} = .157.$$

The sum (1) has the value:

$$\sin 0^\circ \Delta x + \sin 9^\circ \Delta x + \cdots + \sin 81^\circ \Delta x.$$

Here

$\sin 0^\circ =$	.000	$\sin 45^\circ =$	.707
$\sin 9^\circ =$	.156	$\sin 54^\circ =$	.809
$\sin 18^\circ =$	.309	$\sin 63^\circ =$	.891
$\sin 27^\circ =$	.454	$\sin 72^\circ =$	.951
$\sin 36^\circ =$	.588	$\sin 81^\circ =$	.988
	1.507		4.346

and thus we obtain

$$5.853 \times .157 = .92.$$

The sum (3) is seen to have the value

$$6.853 \times 1.57 = 1.08.$$

The area A is found by integration to be $A = 1$.

EXERCISES

In each example the student should draw the graph accurately to a scale of 10 cm. as the unit (using tables of squares, square roots, and reciprocals), and put in the inscribed and the circumscribed rectangles.

1. Compute approximately the area under the curve

$$y = \frac{1}{1 + x^2}$$

for the interval $0 \leqq x \leqq 1$, taking $n = 10$.

Ans. The sum (1) has the value 0.810; the sum (2), the value 0.750. The value of A is found by integration to be 0.785.

2. The same for

$$y = \frac{1}{\sqrt{1 + x^4}}.$$

Here, however, A cannot be found by integration, since the integral cannot be expressed in terms of the functions as yet at our command.

3. The Fundamental Theorem of the Integral Calculus. We are now able to state the theorem on which the whole Integral Calculus rests. It consists essentially in the equality of the expression (2) for A found in § 2 and the expression (1) for A found in § 1:

$$(1) \quad \lim_{n=\infty}\Big[f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x\Big] = \Big[\int f(x)\,dx\Big]_{x=a}^{x=b}.$$

It can be expressed in words as follows:

Fundamental Theorem of the Integral Calculus. *Let $f(x)$ be a continuous function of x throughout the interval $a \leqq x \leqq b$. Divide this interval into n equal parts by the points $x_0 = a, x_1, x_2, \cdots, x_{n-1}, x_n = b$, and form the sum:*

$$f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x.$$

If n now be allowed to increase without limit, this sum will approach a limit; and this limit can be found by integrating the

function $f(x)$ *and taking the integral between the limits* $x = a$ *and* $x = b$*:*

$$\left[\int f(x)\,dx\right]_{x=a}^{x=b}.$$

Expressed as a formula, the theorem is as follows:

$$\lim_{n=\infty}\left[f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x\right] = \left[\int f(x)\,dx\right]_{x=a}^{x=b}.$$

Although the proof of the theorem was geometric, the final result, namely, equation (1), is purely analytic in its character. We may liken the process to that of building a masonry bridge. First a wooden arch is erected. On this are placed the blocks of granite, and when the structure is completed the wooden arch is removed. The bridge is the essential thing, the wood was incidental. And so here the geometrical pictures are but a means to the end, which is an analytical theorem, — the theorem on which the integral calculus rests.

If the expression (4) of § 2 be used for A, the Theorem takes on the form:

$$(2)\quad \lim_{n=\infty}\left[f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x\right] = \left[\int f(x)\,dx\right]_{x=a}^{x=b}.$$

Finally, if (6), § 2, be used, we have:

$$(3)\quad \lim_{n=\infty}\left[f(x_0')\,\Delta x_0 + f(x_1')\,\Delta x_1 + \cdots + f(x_{n-1}')\,\Delta x_{n-1}\right]$$
$$= \left[\int f(x)\,dx\right]_{x=a}^{x=b}.$$

The forms (1) and (2) are the most important in the elementary applications of the calculus.

The Definite Integral. The limit which stands on the left-hand side of equation (1) or (2) or (3) is defined as the *definite integral* of the function $f(x)$, and is written:

$$\int_a^b f(x)\,dx.$$

In distinction from the definite integral, what we have hitherto called the integral of a function, namely,

$$\int f(x)\,dx,$$

is called the *indefinite integral.* Thus the definite integral is, *by definition,* the limit of a sum:

$$(4)\qquad \lim_{n=\infty}\Big[f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x\Big] = \int_a^b f(x)\,dx;$$

i.e. the right-hand side is merely a new notation for the left-hand side. The indefinite integral, on the other hand, is the inverse of a derivative; Chap. IX, § 2. That the definite integral is equal to the indefinite integral taken between the limits $x = a$ and $x = b$:

$$(5)\qquad \int_a^b f(x)\,dx = \Big[\int f(x)\,dx\Big]_{x=a}^{x=b},$$

is precisely the content of the Fundamental Theorem. Equation (5) is a *theorem,* — NOT a definition or notation. The definition is contained in (4).

The integral sign had its origin in the old-fashioned long *s*, the initial letter of *summa*, the integral being thus conceived as a definite integral, the limit of a sum.

Notation for a Sum. A sum of terms, as $u_0 + u_1 + \cdots + u_{n-1}$, is frequently written in the form:

$$(6)\qquad \sum_{k=0}^{n-1} u_k.$$

Similarly, we write:

$$u_1 + u_2 + \cdots + u_n = \sum_{k=1}^{n} u_k.$$

Thus the sums (1) and (3) of § 2 would be abbreviated as

$$\sum_{k=0}^{n-1} f(x_k)\,\Delta x, \qquad\qquad \sum_{k=1}^{n} f(x_k)\,\Delta x.$$

The Fundamental Theorem can now be stated in the form:

$$\lim_{n=\infty}\sum_{k=0}^{n-1} f(x_k)\,\Delta x = \Big[\int f(x)\,dx\Big]_{x=a}^{x=b}. \tag{7}$$

EXERCISE

Prove that, if c lies between a and b,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

4. Volume of a Solid of Revolution. As a first application of definite integrals consider the problem of finding the volume of a solid of revolution bounded by two planes perpendicular to the axis of the solid.

Let a plane curve,

$$y = \phi(x), \tag{1}$$

rotate about the axis of x. It thus generates a surface of revo-

FIG. 90

lution. If, in particular, the curve is a straight line not cutting the axis, we get a cylinder, a cone, or a frustum of a cone. In the first case, the volume is computed from the most elementary considerations, and is found to be equal to the product of the base by the altitude:

$$V = \pi r^2 h.$$

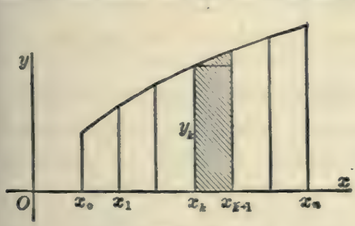


FIG. 91

With the aid of this result, we can solve the general problem. Let the axis of the solid be divided into n equal parts by the points $x_0 = a, x_1, \cdots, x_{n-1}, x_n = b$, and let planes be passed through these points perpendicular to the axis, thus dividing

the solid into slabs. We can approximate to the volumes of these slabs by cylinders of revolution whose bases are the bases of the successive slabs. Thus the k-th slab is replaced by a cylinder whose altitude is Δx and the radius of whose base is

$$y_k = \phi(x_k).$$

Its volume is $\pi y_k^2 \Delta x$, and the sum of the volumes of all such cylinders is

$$\pi y_0^2 \Delta x + \pi y_1^2 \Delta x + \cdots + \pi y_{n-1}^2 \Delta x.$$

It is clear on visualizing the two solids, — the given solid and the one made up of the n cylinders, — that, as n is allowed to grow larger and larger without limit, this sum approaches the volume V of the solid in question as its limit:

$$V = \lim_{n=\infty} [\pi y_0^2 \Delta x + \pi y_1^2 \Delta x + \cdots + \pi y_{n-1}^2 \Delta x].$$

Now, the right-hand side of this equation is precisely of the form of the definite integral (4), § 3, where the function $f(x)$ is chosen as πy^2:

$$f(x) = \pi y^2 = \pi [\phi(x)]^2.$$

Hence

$$(2) \qquad V = \pi \int_a^b y^2 dx.$$

The constant factor, π, has been taken outside of the sign of integration, and this is legitimate, for the factor can be taken outside of the sum, and hence also outside of the limit of the sum.

We have thus obtained a *formulation* of the volume V we set out to compute, as a *definite integral*. But a definite integral can be evaluated, by the Fundamental Theorem, by means of the indefinite integral taken between the proper limits.

Example 1. To find the volume of a cone of revolution. Here, the generating curve (1) is a straight line, Fig. 90, and its equation, if the axis of y is chosen in the simplest manner possible, is

$$y = \lambda x, \qquad \lambda = \frac{r}{h},$$

where h is the altitude of the cone and r is the radius of the base. In this case, $\phi(x) = \lambda x$. Hence, from (2),

$$V = \pi \int_0^h \lambda^2 x^2 \, dx = \pi\lambda^2 \int_0^h x^2 \, dx.$$

To compute the definite integral, find first the indefinite integral:

$$\int x^2 \, dx = \frac{x^3}{3}.$$

This must be taken between the limits 0 and h:

$$\left[\int x^2 \, dx\right]_{x=0}^{x=h} = \left[\frac{x^3}{3}\right]_0^h = \frac{h^3}{3}.$$

Thus
$$V = \pi\lambda^2 \frac{h^3}{3}.$$

On replacing λ by its value, r/h, we have the usual formula for the volume of a cone:

(3)
$$V = \frac{\pi}{3} r^2 h.$$

The student may ask why we did not add a constant in the formula for the indefinite integral:

$$\int x^2 \, dx = \frac{x^3}{3} + C.$$

It would have been correct to do so, — but useless; since the result would have been the same as before:

$$\left[\frac{x^3}{3} + C\right]_0^h = \left(\frac{h^3}{3} + C\right) - (0 + C) = \frac{h^3}{3}.$$

Example 2. To find the volume of a sphere. Here, the equation of the circle, half of which can be taken as forming the curve (1), Fig. 90, is, for the simplest choice of the y-axis,

$$x^2 + y^2 = r^2.$$

Hence
$$y = \phi(x) = \sqrt{r^2 - x^2}.$$

The volume of the sphere is thus seen to be:

$$V = \pi \int_{-r}^{r} (r^2 - x^2)\,dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \tfrac{4}{3}\pi r^3.$$

It would have been allowable to use the property of symmetry and compute by integration only the volume V' of the hemisphere corresponding to positive values of x:

$$V' = \pi \int_{0}^{r} (r^2 - x^2)\,dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{0}^{r} = \tfrac{2}{3}\pi r^3.$$

Remark. It is important to observe that the rôle which geometry plays in this paragraph is quite different from that which it played in § 2. There we were *proving* the Fundamental Theorem. That theorem is, however, essentially *arithmetic* in character, and it is as such — a theorem relating to *numbers* — that it appears in the present application. In this paragraph we have studied a problem of Solid Geometry, and solved it by the aid of the Fundamental Theorem. Fig. 91 is much like Fig. 89, but its rôle is totally different; its object is to enable us to visualize a space figure; and its strips and rectangles are not the strips and rectangles of Fig. 89; they lead to slabs and flat cylinders.

EXERCISES

1. The semi-vertical angle of a cone is 45°, and the altitude is 10 inches. Find its volume by integration, using the *method*, but not the result, set forth above. *Ans.* 1047 c. in.

2. A sphere 12 in. in diameter is cut by a plane which bisects a radius at right angles. Find the volume of the smaller segment. *Ans.* $45\pi = 141.37$ c. in.

3. An ellipse whose axes are of lengths 8 in. and 16 in. respectively rotates about the longer axis. Show that the volume of the ellipsoid of revolution thus generated is 536 c. in.

4. Show that, if the ellipse of the preceding question rotates about the shorter axis, the volume will be twice as great.

5. Find the volume of an ellipsoid of revolution.

Ans. $\frac{4}{3}\pi ab^2$.

6. Find the volume of a frustum of a cone, whose altitude is 8 in. and the radii of whose bases are 12 in. and 16 in.

Ans. 4959 c. in.

7. Find the volume of a frustum of any cone.

Ans. $\frac{1}{3}\pi h(r^2 + rR + R^2)$.

8. A spindle is formed by the rotation of an arch of the curve

$$y = \sin x$$

about its base. Find its volume. *Ans.* 4.935.

9. Show that the volume of any segment of one base of a paraboloid of revolution is one-half that of the circumscribing cylinder.

10. The four-cusped hypocycloid,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

rotates about the axis of x. Find the volume of the solid.

Ans. $\frac{32}{105}\pi a^3$.

11. Find the volume of a segment of the solid of revolution whose surface is generated by the rotation of the catenary

$$y = \tfrac{1}{2}(e^x + e^{-x})$$

about the axis of x, the origin lying in one of the bases.

Ans. $\frac{1}{8}\pi(e^{2h} - e^{-2h} + 4h)$.

12. The hyperbola $x^2 - y^2 = 1$ rotates about the axis of x. Find the volume of the solid bounded by this surface and a cylinder of revolution which has the same axis, and whose radius is 1.

13. A cork-ball 4 in. in diameter, of specific gravity $\frac{1}{4}$, floats in water. How high is the center above the level of the surface of the water?

14. The same for a cork hemisphere of equal radius.

15. A bowl is bounded by the surfaces formed by the rotation of two parabolas about the axis of x. The equation of one of the parabolas is $y^2 = x$. The other parabola cuts this one in the point (1, 1), and it cuts the axis of x at a distance of one-tenth to the left of the origin. If the specific gravity of the material of which the bowl is made is $2\frac{1}{2}$, and if the unit of length is 10 cm., show that the bowl will weigh 392.7 grammes.

16. A torus, or anchor ring, is the solid generated by a circle which rotates about an axis in its plane, not cutting it. Find its volume. *Ans.* $2\pi^2 ab^2$.

5. Area of a Surface of Revolution. The area of a surface of revolution can be computed by integration. Let the coordinates (x, y) of any point on the generating curve,

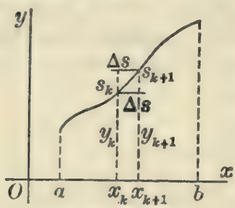


FIG. 92

$$(1) \qquad y = \phi(x),$$

be expressed as functions of the arc, s; cf. p. 135:

$$x = \Phi(s), \quad y = \Psi(s), \quad \alpha \leqq s \leqq \beta.$$

Let the curve be divided into n equal arcs by the points corresponding to $s_0 = \alpha, s_1, s_2, \cdots, s_n = \beta$, and let planes be passed through these points perpendicular to the axis of x, thus dividing the surface into n round strips. Denote the area of the k-th strip by ΔS_k.

Our problem now is to approximate successfully to ΔS_k. This can be done as follows. Suppose the arc (s_k, s_{k+1}), whose length is Δs, to be straightened, without changing its length, into a right line parallel to the axis of x and having one end at the point (x_k, y_k). Let this line rotate about the axis of x, thus generating a right circular cylinder of altitude Δs. It is obvious * that the area of this cylinder, $2\pi y_k \Delta s$, is less than ΔS_k:

$$2\pi y_k \Delta s < \Delta S_k.$$

* *i.e.* it appeals to the intuition as true. Considered mathematically, this assumption (together with the corresponding one just below) is precisely the *geometric axiom* on which the whole evaluation rests.

If, on the other hand, an extremity of the straightened-out arc be taken at the point (x_{k+1}, y_{k+1}), the right circular cylinder now generated will have an area of $2\pi y_{k+1}\Delta s$, which will obviously be greater than ΔS_k:

$$\Delta S_k < 2\pi y_{k+1}\Delta s.$$

Thus we see that ΔS_k is shut in between a minor approximation and a major approximation:

$$(2) \qquad 2\pi y_k \Delta s < \Delta S_k < 2\pi y_{k+1}\Delta s.$$

Write down this double inequality for the successive values of k: $k = 0, 1, \cdots, n-1$:

$$(3) \qquad \begin{array}{c} 2\pi y_0 \Delta s < \Delta S_0 < 2\pi y_1 \Delta s, \\ 2\pi y_1 \Delta s < \Delta S_1 < 2\pi y_2 \Delta s, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 2\pi y_{n-1} \Delta s < \Delta S_{n-1} < 2\pi y_n \Delta s. \end{array}$$

And now add up. The middle column gives precisely the sum of all the partial areas, or S; *i.e.* the quantity we are trying to compute.

The left-hand column has the sum

$$(4) \qquad 2\pi y_0 \Delta s + 2\pi y_1 \Delta s + \cdots + 2\pi y_{n-1}\Delta s;$$

and the right-hand column yields

$$(5) \qquad 2\pi y_1 \Delta s + 2\pi y_2 \Delta s + \cdots + 2\pi y_n \Delta s.$$

These two sums, (4) and (5), are variables depending on n, one of which is always less than the fixed quantity, S; the other, always greater. Moreover, their difference,

$$2\pi y_n \Delta s - 2\pi y_0 \Delta s = 2\pi(B - A)\Delta s,$$

where A and B are the ordinates corresponding to the extremities of the curve, approaches the limit 0 as n increases indefinitely. Hence each of the variables must approach S as its limit, and we have, in particular,

$$(6) \qquad S = \lim_{n=\infty}[2\pi y_0 \Delta s + 2\pi y_1 \Delta s + \cdots + 2\pi y_{n-1}\Delta s].$$

The right-hand side of this equation is the definite integral,

$$2\pi\int_{\alpha}^{\beta} y\,ds.$$

For, on referring to (4), § 3, and observing that here s is the independent variable, we see that, if $f(x)$ be replaced by

$$f(s)=2\pi y = 2\pi\Psi(s),$$

that formula becomes identical with (6) above.

We have thus succeeded in formulating the area in question as a definite integral:

(7) $$S = 2\pi\int_{\alpha}^{\beta} y\,ds.$$

By the Fundamental Theorem, the latter can be evaluated by means of the corresponding indefinite integral, taken between proper limits.*

Example. To find the area of a zone of a sphere. Here, the coordinates of a point (x, y) of the generating circle can be expressed conveniently in terms of the central angle:

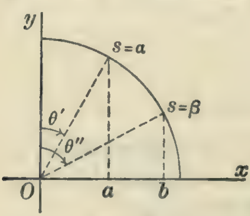


FIG. 93

$$x = r\sin\theta, \qquad y = r\cos\theta, \qquad s = r\theta;$$
$$\theta' \leqq \theta \leqq \theta''; \qquad \alpha = r\theta',$$
$$\beta = r\theta''; \qquad \alpha \leqq s \leqq \beta.$$

The indefinite integral has the value:

$$\int y\,ds = \int (r\cos\theta) r\,d\theta = r^2\sin\theta,$$

and since $x = r\sin\theta$, this can be written as rx.

* It has been tacitly assumed that y increases with x, according to the figure. If y decreased as x increases, the inequality signs in (2) would point the other way. The reasoning would, however, be altogether similar.

If y is increasing in some parts of the interval and decreasing in others, the method can still be adapted without great difficulty; but this case is treated more conveniently by the aid of Duhamel's Theorem, § 8.

Hence

$$\left[\int y\,dx\right]_{x=a}^{x=\beta} = \left[rx\right]_{x=a}^{x=\beta} = rb - ra = rh,$$

where $h = b - a$ denotes the altitude of the zone. We thus obtain as the final result:

(8) $$S = 2\pi rh.$$

From this formula we read off the striking theorem: *The area of a zone of a given sphere depends only on the altitude of the zone, not on its location on the sphere.* For example, if the earth be thought of as a sphere, the area of a zone 10 miles high will be the same, whether one base lies in the equator, or whether one base reduces to a point, — the north pole.

The area of the total surface of a sphere is obtained by setting $h = 2r$:

(9) $$S = 4\pi r^2.$$

Another Form for S. It is usually more convenient to take x as the independent variable, rather than s. Remembering that

$$ds = \sqrt{1 + \frac{dy^2}{dx^2}}\,dx,$$

if s increases with x, we should expect (7) to go over into

(10) $$S = 2\pi \int_a^b y\sqrt{1 + \frac{dy^2}{dx^2}}\,dx,$$

and this is in fact the case.* For, the indefinite integral has the value

$$\int y\,ds = \int y\sqrt{1 + \frac{dy^2}{dx^2}}\,dx,$$

* The truth of this statement may seem to the student self-evident, since all that is needed is to substitute for ds in (7) the above value. But an integrand, like $y\,ds$, is not a product; the differential is merely a part of the notation of the integral, whether it be the indefinite or the definite integral. In the former case we gave a careful proof that the integrand can, nevertheless, be treated as if it were a product, *i.e.* that it obeys the laws of a product; cf. Chap. IX, § 5. The corresponding proof in the latter case is the question here at issue.

and hence, when these functions are taken between corresponding limits, the resulting expressions must have the same value, or

$$\left[\int y\,ds\right]_{s=\alpha}^{s=\beta}=\left[\int y\sqrt{1+\frac{dy^2}{dx^2}}\,dx\right]_{x=a}^{x=b}.$$

But this last expression is equal, by the Fundamental Theorem, to the definite integral (10).

EXERCISES

1. The altitude of a cone of revolution is 12 in., and the diameter of the base is 18 in. Find the lateral area by means of formula (7). *Ans.* $135\pi = 424.0$.

2. The same, using formula (10).

3. Show that the lateral area of a cone of revolution is given by the formula*

$$S = \pi r l,$$

where l denotes the slant height.

4. Find the area of a segment of one base, of a paraboloid of revolution. *Ans.* $\frac{2}{3}\pi\left(\sqrt{m(m+2x)^3} - m^2\right)$.

5. Find the area of an ellipsoid of revolution.

Ans. $2\pi\left(b^2 + \dfrac{ab}{e}\sin^{-1}e\right)$.

* There is little that is edifying in the elementary proof of Solid Geometry for the formula for the *volume* of a cone, based as it is on the evaluation of the volume of a pyramid, since this latter volume is obtained only as the result of a clumsy construction. On the other hand, the formulas for the lateral area of a cone or of a frustum can be obtained with the greatest ease and directness by inscribing regular pyramids or frusta of pyramids, and the student should not fail to visualize these figures.

Similarly, the elementary deduction of formulas (8) and (9) is artificial. But when (9) has once been established, the elementary deduction of the volume of a sphere, considered as the limiting solid enclosed within the various tangent planes, is illuminating.

6. The altitude of a frustum of a cone of revolution is 12 in., and the radii of its bases are 8 in. and 16 in. respectively. Find its lateral area.

7. Show that the lateral area of a frustum of a cone of revolution is given by the formula:

$$S = \pi(r + R)l.$$

Proof, both by the Calculus and by Elementary Geometry.

8. The curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

rotates about the axis of x. Find the total area of the surface generated. *Ans.* $\frac{12}{5}\pi a^2$.

9. The curve

$$r = f(\theta)$$

rotates about the prime vector, $\theta = 0$. Moreover, θ shall lie between p and q, *i.e.* $p \leqq \theta \leqq q$, and p and q shall lie between 0 and π, inclusive. Show that the area of the surface generated is given by the formula

$$(9) \qquad S = 2\pi \int_p^q r \sin\theta \sqrt{r^2 + \frac{dr^2}{d\theta^2}}\, d\theta.$$

10. Find the area of the surface generated by the rotation of the cardioid

$$r = 2a(1 - \cos\theta)$$

about its axis. *Ans.* $\frac{128}{5}\pi a^2$

11. An arc of the equiangular spiral

$$r = ae^{\lambda\theta},$$

corresponding to the range of values of θ from 0 to π, rotates about the prime direction. Find the area of the surface generated.

$$\textit{Ans.} \quad \frac{2\pi\sqrt{1 + \lambda^2}(1 + e^{2\pi\lambda})a^2}{1 + 4\lambda^2}.$$

12. Compute the value of the answer in the preceding problem when $a = 1$ and $\lambda = 1$.

13. Find the area of an anchor ring, or torus.

14. A spherical shell, one foot in diameter, is made of brass, the thickness of which is negligible, and it floats in water. The brass weighs half a pound per square foot. How deep is the lowest point of the shell below the surface of the water? Take 1 cu. ft. of water as weighing 62 lbs.

6. Centre of Gravity of n Particles. Suppose three particles are given, of masses m_1, m_2, and m_3, and suppose they lie on a straight line. Their centre of gravity, G, can be found as follows. Introduce a coordinate, x, on the line, — *i.e.* think of the

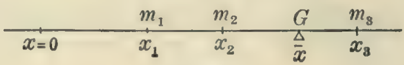


FIG. 94

line as the axis of x, cf. Fig. 94, — and denote the coordinate of the centre of gravity, G, by $\bar{x}$. Then the moment of m_1 about G will be *

$$m_1(\bar{x} - x_1),$$

and the moment of m_3 about G will be

$$m_3(x_3 - \bar{x}),$$

if we understand by *moment* merely the numerical value. But these two moments are of opposite *senses*, — one tending to produce rotation in one direction; the other, in the opposite direction.

It is natural, then, to attach a *sense* to a moment. Thus if we agree that the first of the above moments be taken positive, the second will be defined as negative, and so must now be written

$$m_3(\bar{x} - x_3).$$

* The physical picture we are here assuming is that of a weightless rigid rod, to which the three masses are attached, and which is supported at its centre of gravity, G; the force of gravity acting on the particles. If the rod is at rest, it will stay at rest. The turning effect of each particle, the rod being assumed horizontal, is measured by the moment of the particle about G, as above computed.

The advantage of this generalization of the idea of moment is that we can now write the moment of m_2 as

$$m_2(\bar{x} - x_2),$$

no matter whether, for the particular three masses, G happens to lie to the right or to the left of m_2.

The condition which determines G is that the sum of the numerical moments tending to produce rotation in the one direction is equal to the sum of the numerical moments tending to produce rotation in the opposite direction; or, in terms of our new, algebraic moments, the condition is: *the sum of the algebraic moments is zero.* Hence

$$m_1(\bar{x} - x_1) + m_2(\bar{x} - x_2) + m_3(\bar{x} - x_3) = 0.$$

This equation determines $\bar{x}$. Solving it, we have:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}.$$

The foregoing reasoning is general, applying to the case of n particles on a right line. The result in the general case is:

$$(1) \qquad \bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}.$$

If the n particles are distributed in any manner in a plane, their coordinates being $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, and if the coordinates of G are denoted by $(\bar{x}, \bar{y})$, then $\bar{x}$ will still be given by (1). For, this coordinate would not be changed if, the plane being vertical, each particle were projected vertically on the axis of x, assumed horizontal. Similarly, the y-coordinate is given by the equation

$$(2) \qquad \bar{y} = \frac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{m_1 + m_2 + \cdots + m_n}.$$

If the n particles are situated anywhere in space, their coordinates being $(x_1, y_1, z_1), \cdots, (x_n, y_n, z_n)$, then, besides (1) and (2), there will be a third equation in z:

$$(3) \qquad \bar{z} = \frac{m_1z_1 + m_2z_2 + \cdots + m_nz_n}{m_1 + m_2 + \cdots + m_n}.$$

Example. A granite column 6 ft. high and $1\frac{1}{2}$ ft. in diameter is capped by a ball of the same substance 2 ft. in diameter and stands on a cylindrical granite pedestal 9 in. high and 2 ft. in diameter. How high above the ground is the centre of gravity of the whole post? *Ans.* 4.26 ft.

7. Centre of Gravity of a Solid of Revolution. Consider a homogeneous solid of revolution. Its centre of gravity lies somewhere in the axis of figure, and can be found as follows. Divide the solid into n slabs as set forth in § 4. The centre of gravity of each of these slabs will also lie in its axis, somewhere between the bases, and so, if we denote its coordinate by x'_k, we shall have

(1) $$x_k < x'_k < x_{k+1}.$$

Let the density of the solid be ρ. Then the mass of the k-th slab will be

$$m_k = \rho \Delta V_k,$$

where ΔV_k denotes the volume. Thus $\bar{x}$ is given by formula (1), § 6:

$$\bar{x} = \frac{\rho\Delta V_0 x'_0 + \rho\Delta V_1 x'_1 + \cdots + \rho\Delta V_{n-1} x'_{n-1}}{\rho\Delta V_0 + \rho\Delta V_1 + \cdots + \rho\Delta V_{n-1}},$$

or

(2) $$\bar{x} = \frac{x'_0\Delta V_0 + x'_1\Delta V_1 + \cdots + x'_{n-1}\Delta V_{n-1}}{V}.$$

Here, V is given by (2), § 4:

$$V = \pi\int_a^b y^2\,dx,$$

and is known in the simplest and most important cases from the examples of that paragraph.

We wish to compute the numerator of (2). This can be done as follows. On multiplying (2) through by V, we have:

(3) $$V\bar{x} = x'_0\Delta V_0 + x'_1\Delta V_1 + \cdots + x'_{n-1}\Delta V_{n-1}.$$

Thus the numerator of (2),

(4) $$x'_0\Delta V_0 + x'_1\Delta V_1 + \cdots + x'_{n-1}\Delta V_{n-1},$$

has the *fixed*, but *unknown* value, $V\bar{x}$. *And now we allow n to increase without limit:*

(5) $$V\bar{x} = \lim_{n=\infty}[x_0'\Delta V_0 + x_1'\Delta V_1 + \cdots + x_{n-1}'\Delta V_{n-1}];$$

the reason being that we can compute the right-hand side of (5), as we proceed to show.

Clearly, the volume ΔV_k lies between the volume of the inscribed cylinder, $\pi y_k^2 \Delta x$, and that of the circumscribed cylinder, $\pi y_{k+1}^2 \Delta x$; or

$$\pi y_k^2 \Delta x < \Delta V_k < \pi y_{k+1}^2 \Delta x.$$

Combining this inequality with (1), we have:

(6) $$\pi x_k y_k^2 \Delta x < x_k' \Delta V_k < \pi x_{k+1} y_{k+1}^2 \Delta x.$$

Write out this relation for $k = 0, 1, \cdots, n-1$:

(7) $$\begin{array}{ccccc} \pi x_0 y_0^2 \Delta x & < & x_0' \Delta V_0 & < & \pi x_1 y_1^2 \Delta x, \\ \pi x_1 y_1^2 \Delta x & < & x_1' \Delta V_1 & < & \pi x_2 y_2^2 \Delta x, \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \pi x_{n-1} y_{n-1}^2 \Delta x & < & x_{n-1}' \Delta V_{n-1} & < & \pi x_n y_n^2 \Delta x. \end{array}$$

Now add up. The middle column is precisely the fixed number (4) we are trying to compute. The left-hand column, which is less than (4), is

(8) $$\pi x_0 y_0^2 \Delta x + \pi x_1 y_1^2 \Delta x + \cdots + \pi x_{n-1} y_{n-1}^2 \Delta x,$$

and the right-hand column, which is greater than (4), is

(9) $$\pi x_1 y_1^2 \Delta x + \pi x_2 y_2^2 \Delta x + \cdots + \pi x_n y_n^2 \Delta x.$$

The limit of (8) is the definite integral

(10) $$\lim_{n=\infty}[\pi x_0 y_0^2 \Delta x + \pi x_1 y_1^2 \Delta x + \cdots + \pi x_{n-1} y_{n-1}^2 \Delta x] = \pi \int_a^b xy^2\,dx.$$

The limit of (9) is precisely the same definite integral. Hence the two variables, (8) and (9), approach the same limit, and the fixed number (4), or $V\bar{x}$, always lies between these variables. It follows, then, that the limit of each variable is $V\bar{x}$, or

$$V\bar{x} = \pi \int_a^b xy^2\,dx.$$

From this equation the value of $\bar{x}$ is found:

$$\bar{x} = \frac{\pi \int_a^b xy^2 dx}{V}. \tag{11}$$

In the foregoing proof we have assumed that y increases with x, as shown in Fig. 91, and also that each x_k is positive. If y decreases as x increases, or if y sometimes increases and sometimes decreases; and also if the interval (a, b) lies partly or wholly along the negative axis of x, the result, namely, formula (11), is still true. The proof in this case will be given in § 9.

Example. To find the centre of gravity of a hemisphere. Here we know V to begin with; $V = \frac{2}{3}\pi r^3$. Moreover, as in that example,

$$y^2 = r^2 - x^2.$$

Hence
$$\int_0^r xy^2 dx = \int_0^r x(r^2 - x^2)\, dx = \left[r^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^r = \frac{r^4}{4}.$$

Substituting these values in (11) we find:

$$\bar{x} = \frac{\frac{1}{4}\pi r^4}{\frac{2}{3}\pi r^3} = \frac{3}{8}r.$$

EXERCISES

Find the centre of gravity of each of the following solids.

1. The segment of the ellipsoid of revolution generated by the ellipse
$$9x^2 + 64y^2 = 576,$$
the base being the plane through the centre perpendicular to the axis. *Ans.* $\bar{x} = 3$.

2. The same for the corresponding segment of any ellipsoid of revolution. *Ans.* $\bar{x} = \frac{3}{8}a$.

3. A cone. *Ans.* The point of the axis which is three times as far from the vertex as it is from the base.

4. A segment of one base, of a paraboloid of revolution.

5. The frustum of § 4, Ex. 6.

6. Any frustum of a cone.

Ans. Distance from smaller base, $\frac{h}{4}\,\frac{3R^2+2Rr+r^2}{R^2+Rr+r^2}$.

7. Any segment of one base, of a sphere.

8. The solid corresponding to the curve

$y = \sin x, \qquad 0 \leqq x \leqq \frac{\pi}{2}.\qquad$ *Ans.* $\bar{x} = \frac{\pi}{4} + \frac{1}{\pi} = 1.104.$

9. Show that the centre of gravity of a surface of revolution bounded by two planes perpendicular to the axis is given by the formula

$$\bar{x} = \frac{2\pi\int_a^\beta xy\,ds}{S} = \frac{2\pi\int_a^b xy\sqrt{1+\frac{dy^2}{dx^2}}\,dx}{S}.$$

10. Prove that the centre of gravity of any zone of a sphere lies midway between the bases of the zone.

11. Find the centre of gravity of the lateral surface of a cone of revolution. *Ans.* $\frac{2}{3}h$.

12. Find the centre of gravity of the lateral surface of a segment of a paraboloid.

8. Duhamel's Theorem. We come now to a theorem of fundamental importance in the application of the Calculus to physics and geometry.*

* Both the theorem and the proof will appear to the student, on the first reading, as abstract; they are. After a careful reading of the theorem, he should turn to the application given in § 9 and perceive how this theorem forms the bridge between the physics of the problem and the mathematical formulation. The situation is typical. Duhamel's theorem is, in a great number of similar cases, the connecting link between the physical law which underlies the solution of the problem in hand and the mathematical expression, the definite integral.

The proof of the theorem is illumined by the résumé at the end of this paragraph.

It is important that the student take up Duhamel's Theorem afresh with each new application. He will then soon see that this theorem is an indispensable tool for all problems of this class.

Duhamel's Theorem. *Let*

$$(A) \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

be a sum of positive infinitesimals which approaches a limit when n becomes infinite; and let

$$(B) \qquad \beta_1 + \beta_2 + \cdots + \beta_n$$

be a second sum such that β_k differs from α_k by an infinitesimal of higher order:

$$\lim_{n=\infty}\frac{\beta_k}{\alpha_k} = 1, \qquad \frac{\beta_k}{\alpha_k} = 1 + \epsilon_k, \qquad \beta_k = \alpha_k + \epsilon_k\alpha_k,$$

where ϵ_k is infinitesimal, i.e. $\lim\limits_{n=\infty} \epsilon_k = 0$, k varying in any manner whatever as n increases. Then the sum (B) approaches a limit, and these two limits are equal:

$$\lim_{n=\infty}[\beta_1 + \beta_2 + \cdots + \beta_n] = \lim_{n=\infty}[\alpha_1 + \alpha_2 + \cdots + \alpha_n].$$

In accordance with the hypotheses of the theorem we have:

$$\beta_k = \alpha_k + \epsilon_k\alpha_k.$$

On writing out this equation for the successive values of k, namely, $k = 1, 2, \cdots, n$, and adding these n equations together, we obtain the equation:

$$\begin{aligned}\beta_1 + \beta_2 + \cdots + \beta_n = {} & \alpha_1 + \alpha_2 + \cdots + \alpha_n \\ & + \epsilon_1\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon_n\alpha_n,\end{aligned}$$

and we wish to show that the last line approaches 0 when $n = \infty$. Let η be numerically the largest of the ϵ_k's. Then

$$\left.\begin{array}{l} -\eta \leqq \epsilon_1 \leqq \eta, \\ -\eta \leqq \epsilon_2 \leqq \eta, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ -\eta \leqq \epsilon_n \leqq \eta, \end{array}\right\} \quad \therefore \quad \left.\begin{array}{l} -\eta\alpha_1 \leqq \epsilon_1\alpha_1 \leqq \eta\alpha_1, \\ -\eta\alpha_2 \leqq \epsilon_2\alpha_2 \leqq \eta\alpha_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ -\eta\alpha_n \leqq \epsilon_n\alpha_n \leqq \eta\alpha_n. \end{array}\right\}$$

Hence

$$-(\alpha_1 + \alpha_2 + \cdots + \alpha_n)\eta \leqq \epsilon_1\alpha_1 + \cdots + \epsilon_n\alpha_n \leqq (\alpha_1 + \alpha_2 + \cdots + \alpha_n)\eta.$$

But η approaches 0 when n becomes infinite, and $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ remains finite. Hence the middle term of the double

inequality lies between two variables, each of which is approaching 0 as its limit. This term must, therefore, also approach 0 as its limit, q. e. d.

Résumé. The point of Duhamel's Theorem can be stated as follows. We meet in practice a sum of infinitesimals,

$$(B) \qquad \beta_0 + \beta_1 + \cdots + \beta_{n-1},$$

whose limit we wish to compute. Its terms suggest the terms of a second sum,

$$(A) \qquad \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1},$$

whose limit is a definite integral, the general term being of the form

$$\alpha_k = f(x_k)\,\Delta x.$$

And now we can show, in these cases, that α_k and β_k differ from each other only by a *small percentage of either*, and this percentage furthermore, *even when taken for the most unfavorable pair, approaches* 0 when n becomes infinite.

Under these conditions the two sums (A) and (B), although in general never equal to each other for a single value of n, nevertheless both approach the same limit, and thus the limit of (B) is given as a definite integral.

The underlying thought of Duhamel's Theorem is well brought out by an illustration which Professor Birkhoff uses in presenting this subject. "Suppose," he says, "that I am trying to determine the cost of certain construction, and suppose that I can estimate the individual items, — the cement, the lumber, the paint, the hardware, etc., — each with an error of not more than ten per cent of the number I set down. Then the error in the total estimate will not exceed ten per cent of that estimate. Thus, if the true cost of the cement, — the cost which I ultimately pay, — is β_1 dollars, and if the estimated cost which I set down is α_1 dollars, I am sure that

$$\beta_1 = \alpha_1 + \epsilon_1\alpha_1,$$

where ϵ_1 lies between $+ .10$ and $- .10$:

$$- .10 \leqq \epsilon_1 \leqq + .10.$$

Similarly, for the cost β_2 of the lumber. If I set it down as α_2 dollars, then I am sure that

$$\beta_2 = \alpha_2 + \epsilon_2\alpha_2, \qquad\qquad -.10 \leqq \epsilon_2 \leqq .10.$$

And so on. Finally, the true cost, $\beta_1 + \beta_2 + \cdots + \beta_n$, of the whole construction will differ from the estimated cost, $\alpha_1 + \alpha_2 + \cdots + \alpha_n$, by

$$\epsilon_1\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon_n\alpha_n,$$

and this latter sum lies, as shown in the foregoing proof, between $+10$ per cent and -10 per cent of the estimated cost, $\alpha_1 + \alpha_2 + \cdots + \alpha_n$."

EXERCISES

1. Prove Duhamel's Theorem when the α's and the β's are all negative, the other hypotheses remaining as before.

2. Prove the theorem when the α's and β's are allowed to be positive or negative, but the sum

$$|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$$

remains finite.

Suggestion. The sum $\epsilon_1\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon_n\alpha_n$ does not exceed numerically the sum $|\epsilon_1\alpha_1| + |\epsilon_2\alpha_2| + \cdots + |\epsilon_n\alpha_n|$.

9. **Application.** In § 7 we established the result embodied in formula (11) for the case that y increases as x increases, x being positive. If, however, y decreases as x increases, the relation for ΔV_k will read

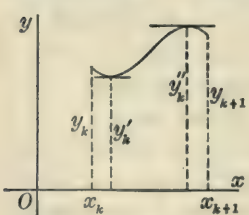


Fig. 95

$$\pi y_{k+1}^2 \Delta x < \Delta V_k < \pi y_k^2 \Delta x,$$

(1) remaining as before. Thus (6) becomes, when x is positive,

$$\pi x_k y_{k+1}^2 \Delta x < x_k' \Delta V_k < \pi x_{k+1} y_k^2 \Delta x.$$

It may also happen that y increases with x in some parts of the interval, and decreases as x increases in others; and all of this may even happen in a subinterval. Moreover, y may be constant in certain parts of the interval (a, b). Let

y'_k denote the smallest value which y attains in the k-th interval, and y''_k, the largest. Then

$$\pi y'^2_k \Delta x \leqq \Delta V_k \leqq \pi y''^2_k \Delta x,$$

and the double inequality becomes

$$\text{(i)} \qquad \pi x_k y'^2_k \Delta x < x'_k \Delta V_k < \pi x_{k+1} y''^2_k \Delta x.$$

On writing out the scheme of relations corresponding to (7), § 7, and adding, we find that the fixed number (4), § 7, which we wish to compute, lies between the two variables

$$\text{(ii)} \qquad \pi x_0 y'^2_0 \Delta x + \pi x_1 y'^2_1 \Delta x + \cdots + \pi x_{n-1} y'^2_{n-1} \Delta x$$

and

$$\text{(iii)} \qquad \pi x_1 y''^2_0 \Delta x + \pi x_2 y''^2_1 \Delta x + \cdots + \pi x_n y''^2_{n-1} \Delta x.$$

Neither of these variables is precisely of the form of the variable in a definite integral, but each suggests the definite integral

$$\text{(iv)} \lim_{n=\infty} [\pi x_0 y_0^2 \Delta x + \pi x_1 y_1^2 \Delta x + \cdots + \pi x_{n-1} y_{n-1}^2 \Delta x] = \pi \int_a^b x y^2 dx.$$

That this definite integral is in fact the value of (4), § 7, can be shown by Duhamel's Theorem. Let

$$\alpha_k = \pi x_k y_k^2 \Delta x, \qquad\qquad \beta_k = x'_k \Delta V_k.$$

Then (i) becomes

$$\pi x_k y'^2_k \Delta x < \beta_k < \pi x_{k+1} y''^2_k \Delta x.$$

On dividing this relation through by α_k, we find:

$$\frac{y'^2_k}{y_k^2} < \frac{\beta_k}{\alpha_k} < \frac{x_{k+1}}{x_k} \cdot \frac{y''^2_k}{y_k^2}.$$

When n becomes infinite, k varying in any manner whatever, each extreme of this double inequality approaches 1 as its limit. Hence the middle term, β_k/α_k, must also approach 1, and the conditions of Duhamel's Theorem are fulfilled. It follows, then, that the two limits,

$$\lim_{n=\infty}[\beta_0 + \beta_1 + \cdots + \beta_{n-1}] = V\bar{x}$$

and

$$\lim_{n=\infty}[\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}] = \pi \int_a^b xy^2 dx$$

are equal, and formula (11) of § 7 is established.

Remarks. 1. If the interval (a, b) lies partly or wholly along the negative axis of x, we can make a change of coordinates:

$$x' = x + C,$$

taking the new origin so far to the left that the interval will now lie along the positive axis. For the new coordinates, the above analysis holds, and hence

$$\bar{x}' = \frac{\pi \int_{a'}^{b'} x'y^2 dx'}{V}.$$

Now, $\pi \int_{a'}^{b'} x'y^2 dx' = \pi \int_a^b (x + C)y^2 dx = \pi \int_a^b xy^2 dx + C\pi \int_a^b y^2 dx.$

The value of the last term is CV, and hence

$$\bar{x}' = \frac{\pi \int_a^b xy^2 dx}{V} + C.$$

On the other hand, $\bar{x}' = \bar{x} + C$.

Hence
$$\bar{x} = \frac{\pi \int_a^b xy^2 dx}{V}, \qquad \text{q. e. d.}$$

The difficulty could also have been met by observing that, if $\bar{y}_k$ be suitably chosen between y'_k and y''_k, ΔV_k can always be written in the form

$$\Delta V_k = \pi \bar{y}_k^2 \Delta x, \qquad y'_k \leqq \bar{y}_k \leqq y''_k.$$

On multiplying (1), § 7, through by this value of ΔV_k we have:

$$\pi x_k \bar{y}_k^2 \Delta x \leqq x'_k \Delta V_k \leqq \pi x_{k+1} \bar{y}_k^2 \Delta x.$$

From this point on the proof proceeds as before, the form of Duhamel's Theorem being that of Ex. 2, § 8.

2. We have tacitly assumed that the bounding curve, $y = \phi(x)$, does not meet the axis of x; for if it did, y_k would not always be different from 0.

Suppose this curve meets the axis in one or both of its end-points. We can then consider a slightly smaller solid, corresponding to a subinterval (a', b') of the interval (a, b):

$$a' \leqq x \leqq b', \qquad a < a' < b' < b.$$

For this solid, the above proof is valid, and we have:

$$\bar{x}' = \frac{\pi \int_{a'}^{b'} xy^2\,dx}{V'}.$$

Now let a' appoach a, and b' approach b. Then $\bar{x}'$ approaches $\bar{x}$, and the right-hand side approaches the right-hand side of (11), § 7.

Such supplementary considerations in the application of Duhamel's Theorem as those of the foregoing remarks occur frequently in practice. They are of the nature of a detail, and since they can always be dealt with in the manner set forth above, we shall not call explicit attention to them again.

10. Centre of Gravity of Plane Areas. Consider a plane sheet of metal, of any shape, the thickness of which is very slight compared with its maximum diameter, like a large sheet of tin, or a piece of gold foil. We are thus led to form the concept of a *material surface*, *i.e.* of matter (mass) spread out in two dimensions. We idealize here just as in geometry, when we form the conception of a point or a line. Actual distributions of matter can, however, as in the cases above cited, come near enough to the ideal cases so that the study of the latter

is sufficient in practice for the treatment of the former. A material surface such as has just been defined is often called a *lamina.*

A material surface is said to be of *uniform density* if the mass, M, of any portion of it is always proportional to the area, A, or

$$M = \rho A,$$

where ρ is the same for all choices of A (but is different for different laminas). The factor ρ is called the *density.*

If, however, this is not the case, the density at an arbitrary point, P, is defined as follows. Enclose P in an arbitrary region of area A, and denote the mass of this piece by M. Then the limit approached by M/A as the boundary of the region shrinks down toward P is called the density, ρ, at P:

$$\lim \frac{M}{A} = \rho.$$

It is understood that even the most remote point of the boundary of the little piece approaches P as its limit.

If we consider an arbitrary region of the lamina, whose area is A and mass M, then

$$M = \bar{\rho} A,$$

where $\bar{\rho}$ denotes the average density. Let ρ' and ρ'' be respectively the least and the greatest values of ρ in the region. Then *

$$\rho' \leqq \bar{\rho} \leqq \rho''$$

The lower signs hold only in case ρ is constant throughout the region. In this case, ρ', ρ'', and $\bar{\rho}$ are all equal to ρ.†

* The truth of this relation is made plausible by the physical meaning of the density. It is possible to prove the relation mathematically. We assume that ρ is a continuous function unless the contrary is stated.

† It is clear how these definitions can be extended to three-dimensional bodies. All that is needed is merely to replace the word *area* by *volume.* Thus a three-dimensional body is of *constant density*, ρ, if $M = \rho V$, no matter what piece be chosen. If, however, this is not the case, the density at a point, P, is defined as

$$\lim \frac{M}{V} = \rho,$$

We shall be concerned for the present only with plane areas of constant density. Let such an area be bounded as in Fig. 89, § 2, and let it be divided as there into strips. If the area of the k-th strip is ΔA_k, its mass will be $m_k = \rho \Delta A_k$, where ρ denotes the density. Let the coordinates of the centre of gravity of this strip be (x'_k, y'_k). Then

$$(1) \qquad x_k < x'_k < x_{k+1}.$$

The x-coordinate of the centre of gravity, G, of the whole area, namely, $\bar{x}$, is given by formula (1), § 6:

$$\bar{x} = \frac{\rho \Delta A_0 x'_0 + \rho \Delta A_1 x'_1 + \cdots + \rho \Delta A_{n-1} x'_{n-1}}{\rho \Delta A_0 + \rho \Delta A_1 + \cdots + \rho \Delta A_{n-1}}$$

or:

$$(2) \qquad \bar{x} = \frac{x'_0 \Delta A_0 + x'_1 \Delta A_1 + \cdots + x'_{n-1} \Delta A_{n-1}}{A}.$$

For the area, A, we have:

$$A = \int_a^b y\,dx,$$

and A is known, in the most interesting and important cases, from the earlier exercises.

We wish to compute the numerator of (2):

$$(3) \qquad x'_0 \Delta A_0 + x'_1 \Delta A_1 + \cdots + x'_{n-1} \Delta A_{n-1}.$$

The solution is similar to that in the case of a solid of revolution.

Let y'_k and y''_k be respectively the least and the greatest ordinates in the k-th strip; cf. Fig. 95. Then

$$(4) \qquad y'_k \Delta x \leqq \Delta A_k \leqq y''_k \Delta x.$$

From (1) and (4) it follows that

$$(5) \qquad x_k y'_k \Delta x < x'_k \Delta A_k < x_{k+1} y''_k \Delta x.$$

where V is the volume of an infinitesimal piece enclosing P. For an arbitrary region of the distribution, whose volume is V and mass M, we have: $M = \bar{\rho} V$, where $\rho' \leqq \bar{\rho} \leqq \rho''$.

On writing out this relation for the successive values of k: $k=0, 1, \cdots, n-1$, and adding, it appears that (3) is greater than

$$(6)\qquad x_0y_0'\Delta x + x_1y_1'\Delta x + \cdots + x_{n-1}y_{n-1}'\Delta x,$$

but less than

$$(7)\qquad x_1y_0''\Delta x + x_2y_1''\Delta x + \cdots + x_ny_{n-1}''\Delta x.$$

Each of these sums suggests the definite integral

$$(8)\qquad \lim_{n=\infty}[x_0y_0\Delta x + x_1y_1\Delta x + \cdots + x_{n-1}y_{n-1}\Delta x] = \int_a^b xy\,dx.$$

That this is, in fact, the value of (3) can be shown by Duhamel's Theorem, on setting

$$\alpha_k = x_ky_k\Delta x, \qquad \beta_k = x_k'\Delta A_k.$$

Relation (5) then becomes:

$$x_ky_k'\Delta x < \beta_k < x_{k+1}y_k''\Delta x.$$

Dividing through by α_k, we have:

$$\frac{y_k'}{y_k} < \frac{\beta_k}{\alpha_k} < \frac{x_{k+1}}{x_k}\frac{y_k''}{y_k}.$$

Hence we infer that

$$\lim_{n=\infty}\frac{\beta_k}{\alpha_k} = 1,$$

and all the conditions of Duhamel's Theorem are fulfilled. It follows, then, that (3) and (8) have the same value, and consequently (2) can be written in the form:

$$(9)\qquad \bar{x} = \frac{\int_a^b xy\,dx}{A}.$$

Example. To find the centre of gravity of a semi-circle. It is obvious that the centre of gravity lies in the axis of symmetry, and so all that is needed is to determine its distance, $\bar{x}$, from the centre, O.

Now, the *abscissa* of G is evidently the same as that of the centre of gravity, G_1, of the upper quadrant, and this is given by (9), where

$$A = \tfrac{1}{4}\pi r^2, \qquad x^2 + y^2 = r^2, \qquad y = \sqrt{r^2 - x^2}.$$

The integral has the value

$$\int_0^r x\sqrt{r^2 - x^2}\,dx = -\frac{(r^2 - x^2)^{\frac{3}{2}}}{3}\Bigg|_0^r = \frac{r^3}{3}.$$

Hence $$\bar{x} = \frac{4r}{3\pi} = .425r.$$

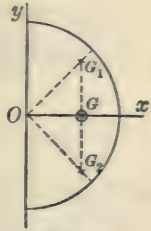


FIG. 96

EXERCISES

Find the centre of gravity of each of the following areas.

1. A semi-ellipse, bounded by the minor axis. *Ans.* $\bar{x} = \dfrac{4a}{3\pi}$.

2. A parabolic segment of one base. *Ans.* $\bar{x} = \frac{3}{5}h$.

3. The area bounded by an arc of a hyperbola and a latus rectum.

4. The same for an ellipse.

5. A right triangle. *Ans.* The intersection of the medians.

6. The area bounded by the lines $x = a$ and $x = b$, and the curves

$$y = f(x), \qquad y = \phi(x),$$

where $$f(x) < \phi(x), \qquad a \leqq x \leqq b.$$

$$\textit{Ans.}\quad \bar{x} = \frac{\displaystyle\int_a^b x(y'' - y')\,dx}{A},$$

where $y' = f(x)$, $y'' = \phi(x)$, and A denotes the area of the lamina. The student should draw a suitable figure and give all the details of the proof of this formula.

7. Any triangle. *Ans.* The intersection of the medians.

8. Define carefully the concept of a *material curve*, or wire whose cross-section is a point. When is such a wire said to be of *uniform density*?

Define the *density* in the general case.

9. Show that the mass of a material curve is

$$M = \int_0^l \rho\, ds = \int_a^b \rho \sqrt{1 + \frac{dy^2}{dx^2}}\, dx,$$

where ρ denotes the density, assumed continuous.

10. The density of a straight line is proportional to the distance from one end: $\rho = cx$. Find its mass. *Ans.* $M = \frac{1}{2} cl^2$.

11. The same, if the density is proportional to the square of the distance from one end.

12. Show that the abscissa of the centre of gravity of a material curve is given by the formula:

$$\bar{x} = \frac{\displaystyle\int_0^l \rho x\, ds}{M},$$

where M denotes the mass of the curve.

13. Under what restrictions can the equation of the last question be replaced by the formula

$$\bar{x} = \frac{\displaystyle\int_a^b \rho x \sqrt{1 + \frac{dy^2}{dx^2}}\, dx}{M}?$$

Find the centre of gravity of each of the following material curves.

14. A straight wire whose density is proportional to the distance from one end. *Ans.* $\bar{x} = \frac{2}{3} l$.

15. The same, if the density is proportional to the square of the distance. *Ans.* $\bar{x} = \frac{3}{4} l$.

16. A uniform semi-circular wire. *Ans.* $\bar{x} = \dfrac{2r}{\pi} = .637r$.

17. Give a qualitative explanation to show why $\bar{x}$ is less than $\frac{1}{2}r$ in the case of the semi-circular lamina, but greater than $\frac{1}{2}r$ in the case of the semi-circular wire.

18. The density of a sphere 12 inches in diameter is proportional to the distance from the centre, and is equal to 8 at the surface. Find its mass.

19. The same for a sphere of radius R, if the density at the surface is c.

20. Find the centre of gravity of a hemisphere cut from the sphere of Ex. 19.

21. The density of a rectangular lamina, 8 by 12 inches, is proportional to the distance from one of the short sides, and is equal to 16 at any point on the other short side. Find the mass of the lamina.

22. The density of an isosceles triangle is proportional to the square of the distance from the base, and is equal to c at the vertex. Find the mass of the triangle.

23. Find the centre of gravity of the rectangle of Question 21.

24. The same for the triangle of Question 22.

25. A uniform wire in the shape of a cardioid.

26. The arc of the equiangular spiral $r = e^\theta$, $0 \leqq \theta \leqq \pi$.

11. Fluid Pressure. Let us consider the problem of finding the pressure of a liquid on a vertical wall. Let the surface be bounded as indicated in the figure and let it be divided into n strips by ordinates that are equally spaced. Denote the pressure on the k-th strip by ΔP_k. Then we can approximate to ΔP_k as follows. Consider the rectangle cut out of this strip by a parallel to the axis of x through the point $(x_k\ y_k)$. The pressure on this rectangle is less than that on the given strip; but we do not yet know how great it is. Still, if we turn the rectangle through 90° about its upper side, the ordinate y_k, we shall obviously have decreased the pressure further. Now the pressure on the rectangle in this new position is readily computed. It is precisely the weight of a column of the liquid

standing on this rectangle as base. The volume of such a column is $(x_k + c)y_k\Delta x$, and if we denote by w the weight of a cubic unit of the liquid, then the weight of the column in question is

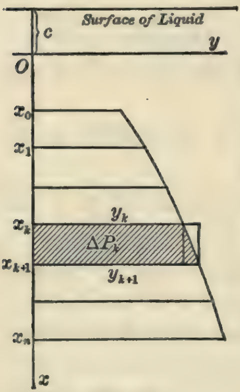


FIG. 97

$$w(x_k + c)y_k\Delta x.$$

This is less than ΔP_k.

In like manner we can find a major approximation by considering the rectangle that circumscribes the given strip and whose altitude is y_{k+1}, and then turning it over on its lower base. The pressure on it in its new position is

$$w(x_{k+1} + c)y_{k+1}\Delta x,$$

and this is larger than ΔP_k. We thus obtain the double inequality:

$$(1) \qquad w(x_k + c)y_k\Delta x < \Delta P_k < w(x_{k+1} + c)y_{k+1}\Delta x.$$

If we write out the relations (1) for $k = 0, 1, \cdots, n-1$:

$$w(x_0 + c)y_0\Delta x < \Delta P_0 < w(x_1 + c)y_1\Delta x,$$
$$w(x_1 + c)y_1\Delta x < \Delta P_1 < w(x_2 + c)y_2\Delta x,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$w(x_{n-1} + c)y_{n-1}\Delta x < \Delta P_{n-1} < w(x_n + c)y_n\Delta x,$$

and add them together, we see that the middle column, namely, the pressure P which we seek to determine, lies between

$$(2) \quad w(x_0 + c)y_0\Delta x + w(x_1 + c)y_1\Delta x + \cdots + w(x_{n-1} + c)y_{n-1}\Delta x$$

and

$$(3) \quad w(x_1 + c)y_1\Delta x + w(x_2 + c)y_2\Delta x + \cdots + w(x_n + c)y_n\Delta x.$$

Finally, allow n to become infinite. The limit of each of the variables (2) and (3) is the definite integral

$$w\int_a^b (x + c)y\,dx.$$

But the pressure P always lies between these variables, and hence it must coincide with their common limit. Thus we see that

$$P = w\int_a^b (x + c)y\,dx. \tag{4}$$

We have deduced our result under the assumption that the ordinates of the bounding curve never decrease as x increases. The formula is true, however, even if this condition is not fulfilled, as we shall show in § 12.

Example 1. To find the pressure on the end of a tank that is full of water.

Here it is convenient to take the axis of y in the surface of the liquid, so that $c = 0$. The equation of the bounding curve is

$$y = B,$$

and thus

$$P = w\int_0^A xB\,dx = wB\frac{x^2}{2}\bigg|_0^A = \frac{wA^2B}{2}.$$

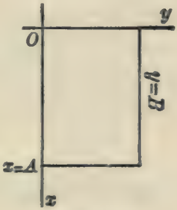


FIG. 98

Now the area of the rectangle is AB, so that, if we write the result in the form

$$P = w \cdot AB \cdot \frac{A}{2},$$

it appears that the total pressure is the same as what it would be if the rectangle were turned through 90° about a horizontal line through its centre of gravity and lying in its surface, and thus supported a column of liquid of height $\frac{1}{2}A$.

EXERCISES

Find the pressure on each of the following vertical surfaces. Take the weight of a cubic foot of water as $62\frac{1}{4}$ lbs.

1. A right triangle with one vertex in the surface, the vertical leg being 18 ft., and the horizontal one, 8 ft., long.

Ans. Nearly 27 tons.

2. The same triangle, if the 8 ft. side lies in the surface.

Ans. Only half as great.

3. A vertical masonry dam in the form of a trapezoid is 200 ft. long at the surface of the water, 150 ft. long at the bottom, and is 60 ft. high. What pressure must it withstand?

Ans. $9337\frac{1}{2}$ tons.

4. A water main 6 ft. in diameter is half full of water. Find the pressure on the gate that closes the main.

Ans. 1120 lb.

5. The same problem, if the main is just full, but the water does not stand above the level of the top. *Ans.* 5280 lb.

6. A cross-section of a trough is a parabola with vertex downward, the latus rectum lying in the surface and being 4 ft. long. Find the pressure on the end of the trough when it is full of water. *Ans* $66\frac{2}{3}$ lb.

7. One end of an unfinished water main 4 ft. in diameter is closed by a temporary bulkhead and the water is let in from the reservoir. Find the pressure on the bulkhead if its centre is 40 ft. below the surface of the water in the reservoir.

Ans. $5\pi = 15.76$ tons.

12. Continuation. In the foregoing paragraph we have assumed that when x increases, y never decreases. If, however, this is not the case, the proof of formula (4) as given in that paragraph breaks down.

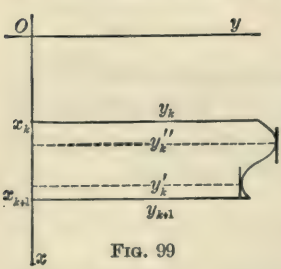


FIG. 99

Let y_k' and y_k'' denote respectively the least and the greatest ordinate in the k-th strip. Then the reasoning of § 11 shows that

$$(1) \qquad w(x_k + c)y_k'\Delta x < \Delta P_k < w(x_{k+1} + c)y_k''\Delta x.$$

Each of the extremes in this double inequality suggests the infinitesimal $w(x_k + c)y_k\Delta x$, together with the corresponding sum

(2) $w(x_0+c)y_0\Delta x + w(x_1+c)y_1\Delta x + \cdots + w(x_{n-1}+c)y_{n-1}\Delta x,$

whose limit is the definite integral

(3) $$\lim_{n=\infty}[w(x_0+c)y_0\Delta x + w(x_1+c)y_1\Delta x + \cdots + w(x_{n-1}+c)y_{n-1}\Delta x] = w\int_a^b (x+c)y\,dx.$$

On the other hand the pressure which we desire to compute is

(4) $$P = \Delta P_0 + \Delta P_1 + \cdots + \Delta P_{n-1} = \lim_{n=\infty}[\Delta P_0 + \Delta P_1 + \cdots + \Delta P_{n-1}].$$

That the limits (3) and (4) have the same value is shown by Duhamel's Theorem, if we set

$$\alpha_k = w(x_k+c)y_k\Delta x, \qquad \beta_k = \Delta P_k,$$

and then divide (1) through by α_k:

$$\frac{w(x_k+c)y_k'\Delta x}{w(x_k+c)y_k\Delta x} < \frac{\Delta P_k}{\alpha_k} < \frac{w(x_{k+1}+c)y_k''\Delta x}{w(x_k+c)y_k\Delta x},$$

or

(5) $$\frac{y_k'}{y_k} < \frac{\beta_k}{\alpha_k} < \frac{x_{k+1}+c}{x_k+c}\cdot\frac{y_k''}{y_k}.$$

The extremes of this double inequality both approach 1 as their limit. Hence

$$\lim_{n=\infty}\frac{\beta_k}{\alpha_k} = 1,$$

and all the conditions of Duhamel's Theorem are satisfied. It follows, then, that for any continuous bounding curve whatsoever, formula (4) of § 11 holds:

$$P = w\int_a^b (x+c)y\,dx.$$

EXERCISES

1. Show that, if the upward sense along the vertical be taken as the positive sense of the axis of x, c still denoting the

distance of the origin below the surface, the formula for the pressure is:

$$P = w\int_a^b (c - x)y\,dx, \qquad a \leqq x \leqq b.$$

2. Work Ex. 6 of § 11, using the result of the last exercise

3. Show that the pressure on the vertical area considered in § 11 is unchanged if the area be rotated about a horizontal line in its surface, which passes through its centre of gravity.

Suggestion. Take the axis of y in the surface of the liquid and use formula (4) of § 11:

$$P = w\int_a^b xy\,dx.$$

Substitute for the integral its value as given by formula (9) of § 10 and interpret the result.

4. Prove the theorem of the preceding question for any vertical plane area.

5. If the surface against which the fluid presses be thought of as rigid, at what height should a horizontal brace be applied, in order that there may be no tendency of the surface to rotate in either direction about the brace?

$$\textit{Ans.}\quad \bar{\bar{x}} = \frac{w\int_a^b (x + c)xy\,dx}{P}.$$

6. If the surface has a vertical axis of symmetry, it would be sufficient to support that point of this axis whose height is given by $\bar{\bar{x}}$. This point is known as the *centre of fluid pressure.* How should you define the centre of fluid pressure when there is no such axis of symmetry?

7. Show that the centre of fluid pressure of the rectangle of § 11, Example, is two-thirds of the way down the rectangle.

8. Find the centre of fluid pressure for the surface of Ex. 3, § 11.

9. The same for Ex. 1, § 11.

13. Volumes. In § 4 the volume of a solid of revolution was computed as a definite integral. It is possible to apply the method to other volumes (in fact, to any volume).

Example. A woodcutter starts to fell a tree 4 ft. in diameter and cuts half way through. One face of the cut is horizontal, and the other face is inclined to the horizontal at an angle of 45°. How much of the wood is lost in chips?

Since the solid whose volume we wish to compute is symmetric, we may confine ourselves to the portion $OABC$. Divide the edge OA into n equal parts and pass planes through these points of division perpendicular to OA. The solid is thus divided into slabs that are nearly prisms; only the face $QRR'Q'$ is not a plane. Let us meet this difficulty by constructing a right prism on PQR as base and with PP' as altitude. Then its volume will be a little greater than that of the actual slab. The solid formed by the n prisms thus constructed differs in volume but slightly from the actual solid, and its volume is seen to approach the volume of the actual solid as its limit when n increases without limit.

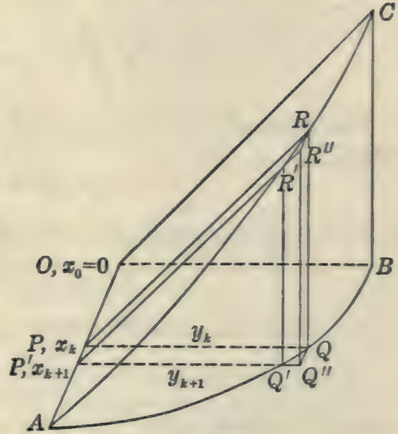


FIG. 100

We will next formulate analytically the volume of the prisms. The base PQR is a 45° right triangle. Let $OP = x_k$ and $PQ = y_k$. Then, by the Pythagorean Theorem,

$$x_k^2 + y_k^2 = 4.$$

Hence the volume of this prism is

$$\tfrac{1}{2} y_k^2 \Delta x = (2 - \tfrac{1}{2} x_k^2)\Delta x,$$

and the volume of the solid we wish to compute is the limit of the sum of all such volumes, or

$$\lim_{n=\infty}\left[\tfrac{1}{2}y_0^2\Delta x + \tfrac{1}{2}y_1^2\Delta x + \cdots + \tfrac{1}{2}y_{n-1}^2\Delta x\right].$$

But this limit is precisely the definite integral

$$\lim_{n=\infty}\left[(2-\tfrac{1}{2}x_0^2)\,\Delta x + (2-\tfrac{1}{2}x_1^2)\,\Delta x + \cdots + (2-\tfrac{1}{2}x_{n-1}^2)\,\Delta x\right]$$
$$= \int_0^2 (2-\tfrac{1}{2}x^2)\,dx.$$

The value of the latter is found by taking the indefinite integral,

$$\int (2-\tfrac{1}{2}x^2)\,dx = 2x - \tfrac{1}{6}x^3,$$

between the limits 0 and 2:

$$2x - \tfrac{1}{6}x^3 \Big|_0^2 = 2\tfrac{2}{3}.$$

The total volume is twice this amount, or $5\frac{1}{3}$ cu. ft.*

* We have based the solution directly on inspection of the figure, and for this reason it is of the utmost importance that the student should visualize the figure clearly. He should make an accurate model, either out of cardboard or by sawing one from a wooden cylinder, and draw the lines PR, $P'R'$, etc. on it with a ruler. He should also reproduce Fig. 100 on paper, both free-hand and with the ruler, drawing first the lines OB, BC, OC, and OA. The angle which OA makes with OB may be chosen arbitrarily within reasonable limits. The line AQB is an arc of an ellipse, whose tangent at A is parallel to OB and whose tangent at B is parallel to OA. These tangents should first be drawn; they can later be erased. The ellipse can then be drawn in free-hand with reasonable accuracy. Next draw PQ parallel to OB, and through P and Q draw lines parallel respectively to OC and BC. Thus R is determined. Similarly for R'. The curve $CRR'A$ can now be drawn in with reasonable accuracy.

A soft pencil well sharpened should be used, so that the curved lines may be easily corrected. It is well to ink in the final figure, and clean it up with art gum. The ultimate result should be a clean piece of mechanical drawing. The student should, however, also be able to make a free-hand drawing with sufficient accuracy to serve satisfactorily the mathematical purposes of the problem. No technical knowledge of Descriptive Geometry on the part of the student is assumed.

EXERCISES*

1. Work the same problem, slicing by planes perpendicular to OB.

2. Show that the volume of a pyramid is one-third the product of the base by the altitude.

Suggestion. Begin with the case of a triangular pyramid, three of whose faces are mutually perpendicular to each other.

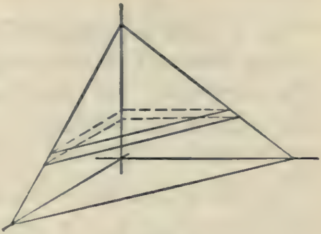


FIG. 101

3. A conoid is a wedge-shaped solid whose lateral surface is generated by a straight line which moves so as always to keep parallel to a fixed plane and to pass through a fixed circle and a fixed straight line; both the line and the plane of the circle being perpendicular to the fixed plane. Find the volume of the solid.**

Ans. $\frac{1}{2}\pi a^2 h$.

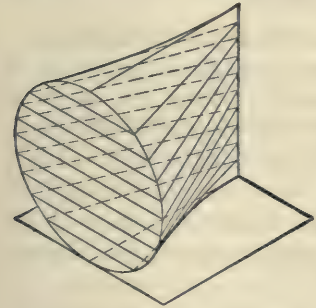


FIG. 102

4. A banister cap is bounded by two equal cylinders of revolution whose axes intersect at right angles in the plane of the base of the cap. Find the volume of the cap. *Ans.* $\frac{8}{3}a^3$.

5. A Rugby foot-ball which has not as yet been inflated is 16 in. long, and the seams form two equal ellipses whose planes are at right angles to each other and whose minor axes are each 8 in. long. Find the volume of the ball, assuming that the

* Draw an accurate figure to represent each solid and the k-th slice. Use symmetry so far as possible.

** The student should draw a working diagram representing one-fourth of the actual solid. This can be done in each of six ways.

leather is so stiff that every cross-section formed by a plane perpendicular to the common major axes of the ellipses is a square. *Ans.* $341\frac{1}{3}$ cu. in.

6. A solid is generated by a variable hexagon which moves so that its plane is always perpendicular to a given diameter of a fixed circle, the centre of the hexagon lying in this diameter, and its size varying so that two of its vertices always lie on the circle. Find the volume of the solid. *Ans.* $2\sqrt{3}a^3$.

7. Show that the volume of an ellipsoid whose semi-axes are of lengths a, b, c is $\frac{4}{3}\pi abc$.

8. Prove Cavalieri's Theorem: If two solids are such that, when cut by a properly chosen system of parallel planes, the area of the cross-section of one of the solids by an arbitrary one of the planes is the same as that of the other solid by the same plane, then their volumes are equal.

9. A glover's needle has, as its cross-section by a plane perpendicular to the axis, an equilateral triangle. Each edge of the needle is a parabola, which meets the axis of the needle in the point of the latter at an angle whose tangent is 0.2; moreover, the axis of the parabola is at right angles to the axis of the needle, and the distance of the vertex of the parabola from the axis of the needle is 0.015 cm. Find the volume of the needle cut off by a plane perpendicular to its axis and distant 2 cm. from its point.

10. A wireless tower is 60 ft. high, and any horizontal cross-section is a square. The vertices of such squares lie in four equal parabolas, whose planes pass through the axis of the tower, and each of the parabolas has its vertex in the upper base of the tower and its axis horizontal. The diagonals of the upper and lower bases of the tower are respectively 2 ft. and 12 ft. Find the volume of the tower.

11. A square hole 6 in. on an edge is cut through a mast 15 in. in diameter, the axis of the hole and the axis of the mast intersecting at right angles. Find the volume of the chips removed.

12. Work the Example of the text, slicing by horizontal planes.

13. A horn is generated by a variable circle, whose plane, always passing through a fixed straight line, rotates about this line. The point of the circle nearest the line describes a quadrant of a fixed circle, whose plane is perpendicular to the line and whose centre lies on the line. The radius of the variable circle is proportional to the angle which its plane makes with the fixed plane through the line and one end of the quadrant. Find the volume of the horn.

Ans. $\frac{1}{24}\pi^2 b^2(4a+3b)$, where a denotes the radius of the quadrant and b the radius of the base of the horn.

Areas of Surfaces

Compute the area of the surface of each of the following solids.

14. The solid of the Example worked in the text.

15. The solid of Ex. 6. **16.** Of Ex. 4. **17.** Of Ex. 5.

14. Moment of Inertia. By the *moment of inertia* of a system of particles about a straight line in space, called the *axis*, is meant the quantity

$$m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2,$$

where m_k denotes the mass of the k-th particle, and r_k, its perpendicular distance from the axis.

Such a sum is commonly represented, as explained in § 3, by the notation

$$\sum_{k=1}^{n} m_k r_k^2, \qquad \text{or} \qquad \sum m_k r_k^2,$$

if it is clear through what values k is to run. Thus we have, as the definition of the moment of inertia, I, of n particles,

$$(1) \qquad I = \sum_{k=1}^{n} m_k r_k^2 = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2.$$

Continuous Distributions. If a body consists of a continuous distribution of matter, like a solid bar or a circular lamina, its

moment of inertia is given by a definition analogous to the above for a system of particles and can be computed by the Calculus. The development of these ideas is the object of the present paragraph.

Physical Meaning. The physical meaning of the moment of inertia of a rigid body is the measure of the resistance which the body opposes, through its inertia, to being rotated with a given constant angular acceleration about the axis; *i.e.* if a given body can be rotated with a fixed angular acceleration about an axis by forces having a constant moment, Q, about that axis, a body of double the moment of inertia would require, for the same angular acceleration, forces having a moment $2Q$ about the axis; a body of one-third the moment of inertia, forces having a moment of $\frac{1}{3}Q$; etc.

An important formula of physics is the following. By a *compound pendulum* is meant any rigid body so mounted that it can turn freely about a horizontal axis; and the body is thought of as executing small oscillations about the axis under the force of gravity. Let T denote the period of the oscillations. Then

$$T = 2\pi\sqrt{\frac{I}{Mgh}}, \tag{2}$$

where I denotes the moment of inertia about the axis; h, the distance of the centre of gravity from the axis; M, the mass; and g, the acceleration of gravity.

The moment of inertia of *plane areas* has important applications in the theory of the bending of beams and girders under a load.*

Two Properties Common to all Moments of Inertia. From the definition embodied in formula (1) follow at once two properties of the moment of inertia of n particles about a given axis.

* The name "moment of inertia" is here misleading, since the conception neither of *inertia* nor of *mass* enters. It is merely the integral appearing in the definition of the moment of inertia, which presents itself in the problem of bending, and the dimension is $[L^4]$, not $[ML^2]$.

First Property. If a system be considered as made up of partial systems, the moment of inertia of the whole system is equal to the sum of the moments of inertia of the separate parts.

Second Property. If some of the particles be removed to a greater distance from the axis, but none brought nearer to it, the moment of inertia is thereby increased.

These properties are fundamental in extending the conception of the moment of inertia to continuous distributions of matter.

Circular Wire. Consider a circular wire of constant or variable density. Let the axis pass through the centre of the circle perpendicular to its plane. Here, all points of the mass, M, in question are at the same distance, r, from the axis, and hence it is natural to define the moment of inertia as

(3) $$I = Mr^2.$$

Circular Disc. Consider next a uniform circular disc of radius a. Let us find its moment of inertia about its centre.* Divide the disc into rings by drawing concentric circles of radii $r_1, r_2, \cdots, r_{n-1}$, where $r_{k+1} - r_k = \Delta r = a/n$, and moreover $r_0 = 0$, $r_n = a$.

The moment of inertia of the whole disc is equal to the sum of the moments of inertia of these rings (First Property). Now the moment of inertia of the k-th ring, ΔI_k, is greater than what it would be if the mass of this ring, ΔM_k, were concentrated along its inner boundary, thus forming a circular wire of radius r_k and mass ΔM_k (Second Property). Since the moment of inertia of this circular wire is, as we have just seen, $r_k^2 \Delta M_k$, we have

$$r_k^2 \Delta M_k < \Delta I_k$$

* In the case of a plane area, or any distribution of matter in a plane, it is customary to speak of its moment of inertia *about a point*, O, of the plane, meaning thereby its moment of inertia about an axis through O perpendicular to the plane.

Similarly, ΔI_k is less than what the moment of inertia would be if the mass, ΔM_k, were spread out along the outer boundary of the ring. Here, the radius of the circular wire is r_{k+1}, and so $\Delta I_k < r_{k+1}^2 \Delta M_k$.

We are thus led to the double inequality

(4) $$r_k^2 \Delta M_k < \Delta I_k < r_{k+1}^2 \Delta M_k$$

Moreover, $\Delta M_k = \rho \Delta A_k$, where ΔA_k denotes the area of the k-th ring. Since

$$\Delta A_k = \pi r_{k+1}^2 - \pi r_k^2 = \pi(r_{k+1} + r_k)(r_{k+1} - r_k)$$

and $r_{k+1} - r_k = \Delta r$, it follows that

$$\Delta A_k = \pi(r_{k+1} + r_k)\Delta r.$$

Hence ΔA_k satisfies the relation:

(5) $$2\pi r_k \Delta r < \Delta A_k < 2\pi r_{k+1} \Delta r.$$

From (4) and (5) we infer:

(6) $$2\pi\rho r_k^3 \Delta r < \Delta I_k < 2\pi\rho r_{k+1}^3 \Delta r.$$

On writing out the double inequality (6) for $k = 0, 1, 2, \cdots, n-1$ and adding the n relations together, the middle column yields precisely I, — the number we are trying to find. The left-hand column is the sum

$$2\pi\rho[r_0^3 \Delta r + r_1^3 \Delta r + \cdots + r_{n-1}^3 \Delta r],$$

and the right-hand column the sum

$$2\pi\rho[r_1^3 \Delta r + r_2^3 \Delta r + \cdots + r_n^3 \Delta r].$$

The limit of each of these sums is one and the same definite integral, and hence

$$I = 2\pi\rho \int_0^a r^3 dr.$$

On evaluating this integral we find: *

* In the statement of the problem we have just solved there is a certain amount of laxness of conception; for hitherto we have insisted on the view that it is a question of a new *definition* each time that we ex-

(7) $$I = \frac{\pi\rho a^4}{2}.$$

The mass of the disc is $M = \pi\rho a^2$. Hence

(8) $$I = \frac{Ma^2}{2}.$$

Radius of Gyration. If the moment of inertia of a body be written in the form:

$$I = Mk^2,$$

k is called the *radius of gyration.* The radius of gyration is defined, then, as $\sqrt{I/M}$. It may be interpreted as follows: if all the mass were spread out uniformly along a circular wire of radius k, the axis passing through the centre of the ring at right angles to its plane, the moment of inertia would still be the same: $I = Mk^2$. The radius of gyration of the above circular plate is $a/\sqrt{2}$.

EXERCISES

Determine the moment of inertia of each of the following bodies.

1. A ring bounded by two concentric circles of radii a and b, about its centre.

Apply the *method*, but not the result, of the example worked in the text. *Ans.* $\frac{1}{2}M(a^2 + b^2)$.

2. A uniform rod, of length l, about one end. *Ans.* $\frac{1}{3}Ml^2$.

3. A uniform rod, $l = 2a$, about its mid-point. *Ans.* $\frac{1}{3}Ma^2$.

tend the notion of the moment of inertia to a new kind of distribution. Yet here we have proposed the problem of *finding*, i.e. *computing*, a moment of inertia not as yet defined. The answer is that we may adopt an alternative view, considering the moment of inertia of a body as a physical constant, having the two properties above mentioned, and then proceed to compute it by comparing it with other distributions for which the moment of inertia is already defined. The first conception is simpler from the standpoint of postulates and axioms; the second brings the moment of inertia into more obvious relation to its physical meaning.

4. A rectangle, of sides $2a$ and $2b$, about a line through the centre and parallel to the sides whose length is $2a$.

Ans. $\frac{1}{3}Mb^2$.

5. An isosceles triangle about the base.

Ans. $\frac{1}{6}Mh^2$, where h denotes the altitude.

6. An isosceles triangle about the median through the vertex. *Ans.* $\frac{1}{6}Ma^2$, where a is half the base.

7. A scalene triangle about a median.

Ans. $\frac{1}{6}Mh^2$, where h is the distance of either vertex from the median.

8. A circular disc about a diameter. *Ans.* $\frac{1}{4}Ma^2$.

9 A semi-circular disc about its bounding diameter.

Ans. $\frac{1}{4}Ma^2$.

10. An ellipse about an axis.

Ans. $\frac{1}{4}Mh^2$, where h denotes the length of the other axis.

11. The segment of a parabola cut off by the latus rectum, about this line.

12. The same for a hyperbola.

13. A uniform circular wire about a diameter. *Ans.* $\frac{1}{2}Ma^2$.

Find the radius of gyration in each of the following cases.

14. A circular disc whose density is proportional to the distance from the centre, about the centre. *Ans.* $k = a\sqrt{\frac{3}{5}}$.

15. A straight wire whose density is proportional to the distance from one end, about that end.

15. Continuation. *Cylindrical Surface.* Consider any distribution of matter spread out over the surface of a cylinder of revolution. The density of the distribution may be variable, and its shape irregular. It may consist in part of material points and curves. Since each point of the distribution is, however, at one and the same distance, r, from the axis, its moment of inertia about the axis of the cylinder shall be defined to be

$$I = Mr^2.$$

Sphere. The moment of inertia of a homogeneous sphere about a diameter can be found in either of two ways. It is

possible to divide the diameter into n equal parts, pass planes through the points of division perpendicular to the diameter, and then approximate, by means of formula (8), § 14, to the moment of inertia of each slab.

Another way is to divide the sphere up into shells by means of a series of cylinders having the diameter of the sphere as their common axis.* Let their radii be $r_1, r_2, \cdots, r_{n-1}$; let $r_0 = 0$ and $r_n = a$, and let (cf. Fig. 103)

$$r_{k+1} - r_k = \Delta r.$$

Denote the moment of inertia of the portion of the sphere included in the k-th shell by ΔI_k; the corresponding mass by ΔM_k. Then ΔI_k is obviously less than what the moment of inertia of this k-th mass would be if it were concentrated on the outer surface of the shell, thus forming a surface distribution:

$$\Delta I_k < r_{k+1}^2 \Delta M_k.$$

Similarly, ΔI_k is greater than the moment of inertia of the same mass when spread out on the inner surface of the shell. Hence we have the double inequality

(1) $r_k^2 \Delta M_k < \Delta I_k < r_{k+1}^2 \Delta M_k.$

Let ρ denote the density of the sphere, and ΔV_k the volume of the k-th shell. Then

$$\Delta M_k = \rho \Delta V_k.$$

It is readily seen that

(2) $$2\pi r_k \cdot 2 y_{k+1} \Delta r < \Delta V_k < 2\pi r_{k+1} \cdot 2 y_k \Delta r,$$

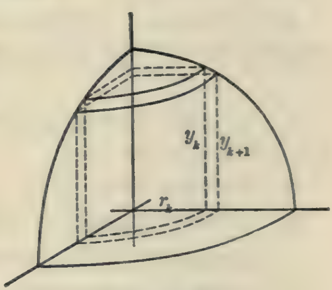


FIG. 103

where

(3) $$r_k^2 + y_k^2 = a^2.$$

From (1) and (2) it follows that

(4) $$4\pi\rho r_k^3 y_{k+1} \Delta r < \Delta I_k < 4\pi\rho r_{k+1}^3 y_k \Delta r.$$

* The analytic details are simpler if the first method is used; but the second method is highly instructive.

From this point on, the procedure is standard. We leave the details to the student. The result is:

$$I = 4\pi\rho \int_0^a r^3 y\,dr, \qquad y = \sqrt{a^2 - r^2}.$$

The indefinite integral is given by Formula 147 of Peirce's *Tables*:

$$\int r^3\sqrt{a^2 - r^2}\,dr = -\left(\tfrac{1}{5}r^2 - \tfrac{2}{15}a^2\right)(a^2 - r^2)^{\frac{3}{2}}.$$

Hence the definite integral has the value $\frac{2}{15}a^5$, and

$$I = \tfrac{8}{15}\pi\rho a^5.$$

Since $M = \frac{4}{3}\pi\rho a^3$, the result can be written in the form:

(5) $$I = \frac{2Ma^2}{5}.$$

EXERCISES

Find the moment of inertia in each of the following cases.

1. A cone of revolution about its axis. Use the method whereby the problem of the sphere was solved in the text.
Ans. $\frac{3}{10}Mr^2$.

2. A sphere about a diameter. Use the first method which was outlined in the text.

3. The same for a cone.

4. A cylinder about its axis. *Ans.* $\frac{1}{2}Ma^2$.

5. A spherical surface about a diameter. *Ans.* $\frac{2}{3}Ma^2$.

6. A conical surface about its axis.

7. The same when the density of the surface is proportional to the distance from the vertex.

8. A homogeneous shell bounded by two concentric spheres of radii a and b.

9. A homogeneous torus, or anchor ring, about its axis.

10. The same for the surface of a torus.

16. A General Theorem. When the moment of inertia of a body about an axis is once known, its moment of inertia about any parallel axis can be found without performing a new integration. The theorem is as follows.

THEOREM. *If the moment of inertia of a body of mass M about an arbitrary axis be denoted by I_0, that about a parallel axis through the centre of gravity by I, then*

$$I_0 = I + Mh^2, \tag{1}$$

where h denotes the distance between the axes.

We will prove the theorem first for a system of particles. Assume a set of Cartesian coordinates (x, y, z), the axis of z being taken as the first axis of the theorem, and then take a second set of Cartesian coordinates (x', y', z') parallel to the first, the origin being at the centre of gravity. Then we have: *

$$I_0 = \sum mr^2 = \sum m(x^2 + y^2),$$

$$I = \sum mr'^2 = \sum m(x'^2 + y'^2).$$

Furthermore,

$$x = x' + \bar{x}, \qquad y = y' + \bar{y}, \qquad z = z' + \bar{z},$$

where $(\bar{x}, \bar{y}, \bar{z})$ is the centre of gravity referred to the (x, y, z) axes. Hence

$$\sum m(x^2 + y^2) =$$

$$\sum m(x'^2 + y'^2) + 2\bar{x}\sum mx' + 2\bar{y}\sum my' + M(\bar{x}^2 + \bar{y}^2).$$

Now $$\sum mx' = 0, \qquad \sum my' = 0.$$

For, recall formula (1) in § 6. Applying that formula to the present system of particles, referred to the (x', y', z')-axes, we see that the abscissa of the centre of gravity, $\bar{x}'$, is:

$$\bar{x}' = \frac{\sum mx'}{M}.$$

* The notation $\sum mr^2$ is an abbreviation for $\sum m_k r_k^2$. Similarly in the ter sums.

But the centre of gravity is at the new origin of coordinates, and so $\bar{x}' = 0$, hence $\sum mx' = 0$. Similarly, $\sum my' = 0$.

It remains only to interpret the terms that are left, and thus the theorem is proven for a system of particles.

If we have a body consisting of a continuous distribution of matter, we divide it up into small pieces, concentrate the mass of each piece at its centre of gravity, form the above sums, and take their limits. We shall have as before $\sum mx' = 0$, $\sum my' = 0$, and hence

$$\sum m(x^2 + y^2) = \sum m(x'^2 + y'^2) + Mh^2,$$

$$\lim \sum m(x^2 + y^2) = \lim \sum m(x'^2 + y'^2) + Mh^2,$$

or

$$I_0 = I + Mh^2, \qquad \text{q. e. d.}$$

The complete details of this last step belong properly in the chapters on double and triple integrals.

EXERCISES

Determine the following moments of inertia.

1. A circular disc about a point in its circumference. *Ans.* $\frac{3}{2}Ma^2$.

2. A uniform rod, of length $2a$, about a point in its perpendicular bisector. *Ans.* $M(\frac{1}{3}a^2 + h^2)$.

3. A rectangle, of sides $2a$ and $2b$, about its centre of gravity. *Ans.* $\frac{1}{3}M(a^2 + b^2)$.

4. The following figures about the axis through the centre of gravity parallel to the lines of the page:

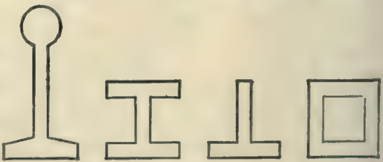


Fig. 104

17. Kinetic Energy of Rotation. When a point, P, describes a circle, the length of its path, s, is connected with the central angle, θ, by the relation

$$s = r\theta.$$

The corresponding relation between its velocity, v, in its path and its angular velocity, ω, is found by differentiation:

$$\frac{ds}{dt}=r\frac{d\theta}{dt}, \qquad \text{or} \qquad v=r\omega.$$

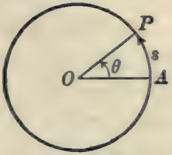


FIG. 105

If a system of particles, $m_1, m_2, \cdots, m_n$, rigidly connected, is rotating about a fixed axis, each particle describes a circle about that axis, and their angular velocities will all have one and the same value, ω. For, consider two of the particles, as m_1 and m_2. The two planes through the axis and these particles always make the same angle with each other. Hence the rate at which one of the planes revolves is the same as that at which the other revolves.

The kinetic energy, E, of the system will be:

$$E=\sum_{k=1}^{n}\frac{m_k v_k^2}{2}=\frac{m_1v_1^2}{2}+\frac{m_2v_2^2}{2}+\cdots+\frac{m_nv_n^2}{2}.$$

Since $v_k=r_k\omega$, where r_k denotes the distance of the k-th particle from the axis, it follows that

$$E=\sum_{k=1}^{n}\frac{m_k r_k^2\omega^2}{2}=\frac{\omega^2}{2}\sum_{k=1}^{n}m_k r_k^2.$$

This last sum is precisely the moment of inertia, I, of the system about the axis, and hence we have the result:

(1) $$E=\tfrac{1}{2}I\omega^2.$$

Since the moment of inertia of a continuous distribution of matter can be expressed as the limit of the moment of inertia of a sum of particles, it follows in the same way that the kinetic energy of any rigid body which is rotating about a fixed axis is given by the formula (1).

Example. To find the kinetic energy of a compound pendulum consisting of a uniform rod, of length l and mass M, mounted so that it can oscillate about one end.

Here, $I=Ml^2/3$, § 14, Ex. 3. Hence

$$E=\tfrac{1}{6}Ml^2\omega^2.$$

The examples of §§ 14–16 afford numerous illustrations of the theorem of this paragraph.

18. The Attraction of Gravitation. Sir Isaac Newton discovered the law of universal gravitation. This law asserts that any two particles of matter in the universe attract each other with a force proportional to their masses and inversely proportional to the square of the distance between them:

$$(1) \qquad f \propto \frac{mm'}{r^2}, \qquad f = K\frac{mm'}{r^2},$$

where K is a physical constant.*

By means of the Calculus we can compute the force with which bodies consisting of a continuous distribution of matter attract one another. Let us determine the force which a uniform rod of mass M exerts on a particle of mass m situated in its own line. Divide the rod up into n equal parts and denote the attraction of the k-th segment by ΔA_k. The mass of this segment is $\rho\Delta x$, where ρ denotes the density of the rod. Now if this whole mass $\rho\Delta x$ were concentrated at the nearer end, its attraction would be greater than ΔA_k; and similarly, if it were concentrated at the further end, its attraction would be less. Hence

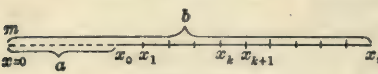


FIG. 106

$$(2) \qquad K\frac{m\rho\Delta x}{x_{k+1}^2} < \Delta A_k < K\frac{m\rho\Delta x}{x_k^2}.$$

Write out the double inequality (2) for $k = 0, 1, \cdots, n-1$ and add the n relations thus resulting. The middle sum, $\Sigma\Delta A_k = A$, is the attraction we wish to compute, and it is seen to lie between two variables, each of which has for its limit the definite integral

$$\int_a^b K\frac{m\rho\, dx}{x^2}.$$

* Called the *gravitational constant*. Its value is

$$6.5 \times 10^{-8}\ \text{cm}^3\ \text{sec}^{-2}\ \text{gr}^{-1}.$$

Hence

$$A = Km\rho\int_a^b \frac{dx}{x^2} = Km\rho\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Km\rho(b-a)}{ab}.$$

The result may be written in the form

$$A = K\frac{mM}{ab},$$

and thus it appears that the rod attracts with the same force as a particle of like mass situated, not at the centre of gravity of the rod, but at a distance from m equal to the geometric mean of the distances a and b of the ends of the rod.

Secondly, suppose the particle were situated in a perpendicular bisector of the rod. Divide the rod as before and consider the attraction of the k-th segment. We must now, however, resolve this force into two components, one perpendicular, the other parallel to the rod. The latter components annul each other for reasons of symmetry, and it is only the sum of the former that we need consider further. We may confine ourselves, moreover, to half the rod and multiply the final result by 2. Let half the rod be divided as shown in the figure, and let the component of the attraction due to the k-th piece be denoted by ΔF_k. Then

$$K\frac{m\rho\,\Delta x}{r_{k+1}^2}\cos\phi_{k+1} < \Delta F_k < K\frac{m\rho\,\Delta x}{r_k^2}\cos\phi_k,$$

and hence we infer by the usual method of reasoning that

$$F = Km\rho\int_0^a \frac{\cos\phi\,dx}{r^2}.$$

Here $\quad r^2 = h^2 + x^2, \quad \cos\phi = \dfrac{h}{\sqrt{h^2+x^2}},$

and so we have, for the total attraction,

$$A = 2F = 2Km\rho h\int_0^a \frac{dx}{\sqrt{(h^2+x^2)^3}}.$$

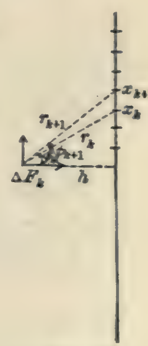


FIG. 107

From Peirce's *Tables*, Formula 138,

$$\int \frac{dx}{\sqrt{(h^2+x^2)^3}} = \frac{x}{h^2\sqrt{h^2+x^2}},$$

and consequently

$$A = \frac{2\,Km\rho}{h}\cdot\frac{x}{\sqrt{h^2+x^2}}\bigg|_0^a = \frac{2\,Km\rho a}{h\sqrt{h^2+a^2}} = K\frac{mM}{h\sqrt{h^2+a^2}}.$$

EXERCISES

Compute the following attractions.

1. A rod whose density varies as the distance from one of its ends, on a particle in its own line.

$$\textit{Ans.}\quad \frac{2\,KmM}{(b-a)^2}\left[\log\frac{b}{a} - \frac{b-a}{b}\right].$$

2. A semi-circular wire, on a particle at its centre.

$$\textit{Ans.}\quad \frac{2\,KmM}{\pi a^2}.$$

3. The same wire, on a particle in the circumference situated symmetrically as regards the wire, and not on the wire.

$$\textit{Ans.}\quad \frac{KmM}{\pi a^2}\log\tan\frac{3\pi}{8}.$$

4. A circular disc, on a particle in the perpendicular to the disc at its centre.

$$\textit{Ans.}\quad \frac{2\,KmM}{a^2}\left[1 - \frac{h}{\sqrt{h^2+a^2}}\right].$$

5. A rod AB, on a particle situated at O in a perpendicular OB at one end.

Ans. A force of $\dfrac{2\,KmM}{hl}\sin\frac{1}{2}AOB$, making an angle of $\frac{1}{2}AOB$ with OB.

6. A rectangle, on a particle in a parallel to two of the sides through the centre.

7. A homogeneous hemispherical shell on a particle at the centre.

For further simple problems in attraction cf. Peirce, *Newtonian Potential Function.*

19. Extension of the Definition of a Definite Integral. In laying down the definition of the definite integral as the limit of the sum (1) or (3) or (5) in § 2, we assumed that a is less than b: $a < b$. The definition can, however, be extended, without modification in its form, to the case that b is less than a. We now have the interval $b \leqq x \leqq a$, and we divide it as before by the points $x_0 = a, x_1, x_2, \cdots, x_n = b$ into n subintervals. Let

$$\Delta x = \frac{b-a}{n}, \qquad \text{or} \qquad \Delta x_k = x_{k+1} - x_k.$$

Then Δx, or Δx_k, will be *negative;* its numerical value will still be the length of the subinterval in question. We will assume, to begin with, that $f(x)$ is positive. Each of the sums (1), (3), and (5) of § 2 will now approach $-A$ as its limit. We define the definite integral as the limit of any one of these sums, and thus

$$\int_a^b f(x)\,dx = -A.$$

On the other hand, by § 3,

$$A = \int_b^a f(x)\,dx,$$

the present a and b being respectively the b and a of that paragraph. Hence we see that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \tag{1}$$

This relation is obviously also true when a is less than b.

Finally, if b is very near to a, the integral is numerically very small, and we see that

$$\lim_{b \doteq a} \int_a^b f(x)\,dx = 0.$$

It is, therefore, natural to lay down the definition:

$$\int_a^a f(x)\,dx = 0.$$

Let $f(x)$ be continuous in the interval $\alpha \leqq x \leqq \beta$, and let a, b, and c be any three points of this interval. Then it is readily shown that

$$(2) \qquad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

or

$$(3) \qquad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^a f(x)\,dx = 0.$$

If $f(x)$ is negative, we shall understand by A the negative of the area bounded by the curve, the axis of x, and the two extreme ordinates. If $f(x)$ changes sign, we shall mean by A the algebraic sum of the areas which lie above the axis of x, taken positively, and the areas which lie below that axis, taken negatively. The definite integral is defined as before, as the limit approached by the sum (1), (3), or (5) of § 2, or the corresponding sum formed for the case of this paragraph. The relations (1), (2), and (3) of this paragraph continue to hold. Moreover, the Fundamental Theorem of the Integral Calculus, § 3, continues to hold for the extended definition.

Example. Consider the integral: $\int_0^{2\pi} \sin x\,dx$. Geometrically, it represents the algebraic sum of two arches of the sine curve, one taken as positive, the other as negative. Its value is therefore 0.

Now, evaluate by means of the indefinite integral:

$$\left[\int \sin x\,dx\right]_0^{2\pi} = -\cos x\Big|_0^{2\pi} = 0,$$

and the results agree.

20. Work Done by a Variable Force. If a force, F, constant in magnitude and always acting along a fixed line AB in the

same sense, be applied to a particle,* and if the particle be displaced along the line in the direction of the force, the work done by the force on the particle is defined in elementary physics as

$$W = Fl,$$

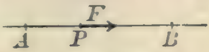


FIG. 108

where l denotes the distance through which the particle has been displaced.

Suppose, however, that the force is variable, but varying continuously and always acting along the same fixed line. How shall the work now be defined?

Let a coordinate be assumed on the line; *i.e.* think of the line as the axis of x. Let the particle be displaced from $A: x = a$ to $B: x = b$, and let $a < b$. Let F, to begin with, always act in the direction of the positive sense along the axis. Then

$$F = f(x),$$

where $f(x)$ denotes a positive continuous function of x.

Divide the interval (a, b) up into n equal parts by the points $x_1, x_2, \cdots, x_{n-1}$, and let $x_0 = a$, $x_n = b$. Then, if

$$x_{k+1} - x_k = \Delta x = (b - a)/n,$$

the work, ΔW_k, done by the force in displacing the particle through the k-th interval ought, in order to correspond to the general physical conception of work, to lie between the quantities

$$F'_k \Delta x \qquad \text{and} \qquad F''_k \Delta x,$$

where F'_k and F''_k denote respectively the smallest and the largest values of $f(x)$ in this interval.** We have, then:

$$(1) \qquad F'_k \Delta x \leqq \Delta W_k \leqq F''_k \Delta x.$$

* Or, more generally, to one and the same point P of a rigid or deformable material body.

** This statement is pure physics. It is the physical axiom on which the generalization of the definition of *work* is based. More precisely, it is one of two physical axioms, the other being that the total work, W, for the complete interval is the sum of the partial works, ΔW_k, for the sub intervals.

On writing out the double inequality (1) for $k=0, 1, \cdots, n-1$ and adding the n relations thus resulting together, we find that $W=\Sigma\Delta W_k$ lies between the two sums

$$(2)\qquad F_0'\Delta x+F_1'\Delta x+\cdots+F_{n-1}'\Delta x,$$

$$(3)\qquad F_0''\Delta x+F_1''\Delta x+\cdots+F_{n-1}''\Delta x.$$

Each of these sums suggests the sum

$$(4)\qquad f(x_0)\Delta x+f(x_1)\Delta x+\cdots+f(x_{n-1})\Delta x,$$

whose limit is the definite integral,

$$(5)\qquad \lim_{n=\infty}[f(x_0)\Delta x+f(x_1)\Delta x+\cdots+f(x_{n-1})\Delta x]=\int_a^b f(x)\,dx.$$

That W is in fact equal to this integral:

$$(6)\qquad W=\int_a^b f(x)\,dx,$$

follows from Duhamel's Theorem if we set

$$\alpha_k=f(x_k)\Delta x, \qquad \beta_k=\Delta W_k.$$

For then

$$W=\beta_0+\beta_1+\cdots+\beta_{n-1}$$
$$=\lim_{n=\infty}[\beta_0+\beta_1+\cdots+\beta_{n-1}],$$

and on dividing (1) through by α_k, we have

$$\frac{F_k'}{f(x_k)}\leqq\frac{\beta_k}{\alpha_k}\leqq\frac{F_k''}{f(x_k)}.$$

Hence

$$\lim_{n=\infty}\frac{\beta_k}{\alpha_k}=1,$$

and all the conditions of the theorem are satisfied.

If the force F acts in the direction opposite to that in which the point of application is moved, we extend the definition and say that negative work is done. For the case that F is constant, the work is now defined as follows:

$$(7)\qquad W=F(b-a).$$

Here, F is to be taken as a negative number equal numerically to the intensity of the force.

Thus (7) is seen to hold in whichever direction the force acts, provided that $a < b$. Will (7) still hold if $b < a$? It will. There are in all four possible cases:

$$(i)\ ++ \qquad (ii)\ -- \qquad (iii)\ +- \qquad (iv)\ -+$$

In cases (*i*) and (*ii*) the force overcomes resistance, and positive work is done. In cases (*iii*) and (*iv*) the force is overcome, and negative work is done. Hence (7) holds in all cases.

It is now easy to see how the definition of work should be laid down when F varies in any continuous manner. The considerations are precisely similar to those which led to equation (6), and that same equation is the final result in this, the most general, case:

$$W = \int_a^b f(x)\,dx.$$

Example. To find the work done in stretching a wire.

Let the natural (or unstretched) length of the wire be l, the stretched length, l'. Then the tension, T, is given by *Hooke's Law:*

$$T = \lambda \frac{l' - l}{l},$$

where λ is independent of l and l', and is known as *Young's Modulus.*

Let the wire, in its natural state, lie along the line OA, and let it, when stretched, lie along OB, OP being an arbitrary intermediate position. Let x be measured from A, and let $x = h$ at B. Then

$$T = \lambda \frac{x}{l}$$

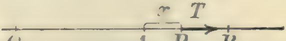


FIG. 109

and
$$W = \int_0^h \lambda \frac{x}{l}\,dx = \frac{\lambda}{l}\int_0^h x\,dx = \frac{\lambda h^2}{2l}.$$

EXERCISES

1. Assuming the sun to attract with a force inversely proportional to the square of the distance from its centre to the particle, find the work done by the sun on a meteor when the latter moves along a straight line passing through the centre of the sun, from an initial distance R to a final distance r.

2. A uniform rod attracts a particle in its own line. Find the work done when the particle moves along a segment of the line, which has no point in common with the rod.

3. The corresponding exercise for the rod of Ex. 1, § 18.

4. A hole is bored through the centre of the earth. It can be shown that the force of attraction at any point of the hole is proportional to the distance from the centre. A stone weighing 1 lb. is dropped into the hole. Find the work done when it reaches the centre.

21. Mean Values. Let the function $y = f(x)$ be continuous in the interval $a \leqq x \leqq b$. Divide the interval into n equal parts by the points $x_0 = a, x_1, x_2, \cdots, x_n = b$ and let $y_k = f(x_k)$. The *average*, or *mean*, of the n values $y_0, y_1, \cdots, y_{n-1}$ is

$$B_n = \frac{y_0 + y_1 + \cdots + y_{n-1}}{n}.$$

Let n increase without limit. Then B_n approaches a limit, B, which can be found as follows. Let $\Delta x = (b-a)/n$ and write B_n in the form:

$$B_n = \frac{y_0 \Delta x + y_1 \Delta x + \cdots + y_{n-1}\Delta x}{b-a}.$$

Then

$$\lim_{n=\infty} B_n = \frac{\lim\limits_{n=\infty}[y_0\Delta x + y_1 \Delta x + \cdots + y_{n-1}\Delta x]}{b-a}.$$

The value of the numerator is the definite integral of the function y. Hence

$$(1) \qquad B = \frac{\int\limits_a^b f(x)\,dx}{b-a}.$$

This limit, B, is called the *mean value* of the function $f(x)$ for the interval (a, b). It obviously lies between the largest and the smallest values of the function. Since $f(x)$ is continuous, there will be at least one value X of x in the interval (a, b), for which $f(x)$ is equal to B:

$$B = f(X).$$

Equation (1) can now be written in the form

$$\int_a^b f(x)\,dx = (b-a)f(X), \qquad a < X < b. \tag{2}$$

This equation is known as the *Law of the Mean* for integrals. Geometrically, equation (2) says that the area under the curve is equal to the area of a rectangle having the same base and a suitably chosen altitude intermediate between the largest and the smallest ordinate in the interval.

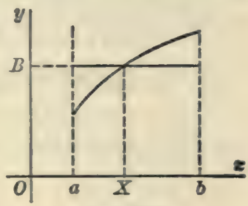


FIG. 110

Equation (2) can also be written in the form

$$\int_a^{a+h} f(x)\,dx = hf(a+\theta h), \quad 0 < \theta < 1. \tag{3}$$

Here, $h = b - a$ and θ is a suitably chosen number intermediate between 0 and 1.

Remark. It should be observed that the average value of the ordinates of a curve in general changes when a new independent variable is introduced. Thus for the quadrant of a circle,

$$y = \sqrt{a^2 - x^2}, \qquad 0 \leqq x \leqq a,$$

the mean value, when x is taken as the independent variable, is

$$B = \frac{\frac{1}{4}\pi a^2}{a} = \frac{\pi a}{4}.$$

But if the angle at the centre is taken as the independent variable, the coordinates of a point on the circle being expressed in the form:

$$x = a\sin\theta, \qquad y = a\cos\theta.$$

then

$$B' = \frac{\int_0^{\frac{\pi}{2}} y\,d\theta}{\frac{\pi}{2}} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} a\cos\theta\,d\theta = \frac{2a}{\pi}\sin\theta\Big|_0^{\frac{\pi}{2}} = \frac{2a}{\pi}.$$

EXERCISES

1. Find the mean value of the positive ordinates of the parabola $y^2 = 2mx$ between the vertex and the latus rectum, when x is taken as the independent variable. *Ans.* $\frac{2}{3}m$.

2. The same, when y is taken as the independent variable. *Ans.* $\frac{1}{2}m$.

3. Show that the average of the $n+1$ ordinates $y_0, y_1, \cdots, y_n$ approaches the same limit, B, when n becomes infinite, as was obtained by considering only the first n y's, namely, $y_0, y_1, \cdots, y_{n-1}$.

22. Numerical Computation. Simpson's Rule. If we wish actually to compute the area under a curve numerically, we can make an obvious improvement on the method of inscribed rectangles by using trapezoids, as shown in Fig. 111. We begin as before by dividing the interval (a, b) into n equal parts, and we denote the length of each part by h. The area of the k-th trapezoid is

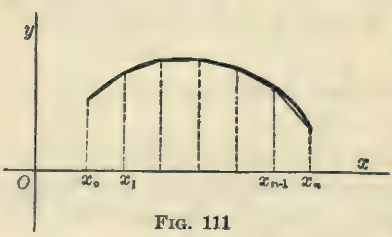


FIG. 111

$$\tfrac{1}{2}(y_k + y_{k+1})h$$

and hence the approximation thus obtained is

$$A_1 = [\tfrac{1}{2}(y_0 + y_n) + (y_1 + y_2 + \cdots + y_{n-1})]h.$$

It is obviously the arithmetic mean of the sum of the areas of the inscribed, and that of the circumscribed, rectangles.

This formula is known as the *Trapezoidal Rule.* If the curve is concave downward, as in Fig. 111, A_1 is too small.

Again, if we take n as an even integer and draw tangents at

the points (x_1, y_1), $(x_3, y_3), \cdots (x_{n-1}\, y_{n-1})$, we get some trapezoids, as shown in the figure, the area of any one being $2y_k h$, where k is odd. Hence

$$A_2 = 2h[y_1 + y_3 + \cdots + y_{n-1}]$$

is an approximation which is too large, and

$$A_1 < A < A_2.$$

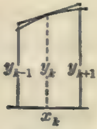


FIG. 112

If the curve is concave upward, the inequalities must be reversed.

Finally, a still closer approximation may be obtained by using arcs of parabolas instead of straight lines. If we make the parabola

$$y = a + b(x - x_k) + c(x - x_k)^2$$

go through three successive points, (x_{k-1}, y_{k-1}), (x_k, y_k), (x_{k+1}, y_{k+1}), it will follow the arc of the curve more closely in between than the broken lines or the tangents of the preceding approximation do.* Now the area under the parabolic arc is

$$\int_{x_k-h}^{x_k+h}[a + b(x - x_k) + c(x - x_k)^2]\,dx =$$

$$ax + b\frac{(x - x_k)^2}{2} + c\frac{(x - x_k)^3}{3}\Bigg|_{x_k-h}^{x_k+h} = 2ah + \frac{2ch^3}{3},$$

and it remains to determine a and c from the above conditions:

$$\begin{aligned} x &= x_k, & y_k &= a; \\ x &= x_k + h, & y_{k+1} &= a + bh + ch^2; \\ x &= x_k - h, & y_{k-1} &= a - bh + ch^2. \end{aligned}$$

Hence $\qquad a = y_k, \qquad 2ch^2 = y_{k-1} - 2y_k + y_{k+1}.$

* The statement is based on the assumption that the curve is smooth, and not crinkly. More precisely, if the function $f(x)$ and its derivatives of the first four orders are continuous, and if the fourth derivative is numerically $\leq M$ throughout the interval, the error will not exceed

$$\frac{(b-a)^5 M}{45 n^4}.$$

Proof by means of Taylor's Theorem with the Remainder. Chapter XIV, § 22

Thus the area under the parabolic arc is seen to have the value
$$\tfrac{1}{3}h(y_{k-1}+4y_k+y_{k+1}).$$
Adding these areas for $k=1, 3, \cdots n-1$, we get a new approximation:
$$A_3=\tfrac{1}{3}h[y_0+y_n+2(y_2+y_4+\cdots y_{n-2})+4(y_1+y_3+\cdots+y_{n-1})].$$
This formula is known as *Simpson's Rule.*

If we set
$$u=y_0+y_n,$$
$$v=y_1+y_3+\cdots+y_{n-1}, \qquad w=y_2+y_4+\cdots+y_{n-2},$$
we have: $\quad A_1=\tfrac{1}{2}h(u+2v+2w), \qquad A_2=2hv,$
$$A_3=\tfrac{1}{3}h(u+4v+2w).$$
It turns out that
$$A_3=\tfrac{2}{3}A_1+\tfrac{1}{3}A_2.$$

*Example.** Consider $\displaystyle\int_1^2\frac{dx}{x}$, and let $n=10$. Then $h=.1$ and
$$u=1.5, \qquad v=3.459\,539\,4, \qquad w=2.728\,174\,6.$$
Hence $A_1=.693\,771, \qquad A_2=.691\,908, \qquad A_3=.693\,150.$

The value of the integral is (*Tables*, p. 109):
$$\log 2=.693\,147.$$

Thus A_1 differs from the true value by less than 7 parts in about 7000, or one tenth of one per cent. A_2 differs by about 12 parts in 7000; while A_3 is in error by less than 3 parts in 600,000, or 1 part in 200,000.

EXERCISES

1. Compute $\displaystyle\int_0^1 e^x\,dx$, taking $n=10$, and compare the result with that obtained by integration. Note the tables on pp. 120, 121 of the *Tables.*

* These figures are taken from Gibson's *Elementary Treatise on the Calculus*, p. 331, to which the student is referred for further examples. A more extended treatment of the subject of this paragraph will be found in Goursat-Hedrick, *Mathematical Analysis*, vol. 1, § 100.

2. Compute
$$\int_1^2 \frac{dx}{\sqrt{1+x^4}}.$$

3. Obtain an approximate formula for the volume of a cask whose bung diameter is a, head diameter b, and length l.

Ans. $\frac{1}{60}\pi l[8a^2+4ab+3b^2]$.

4. Solve the preceding problem, using an ellipse as the generating curve. *Ans.* $\frac{1}{12}\pi l[2a^2+b^2]$.

5. If, in questions 3 and 4, a is only slightly greater than b, the formula may be replaced by the simpler one:

$$\frac{\pi al(a+2b)}{12}.$$

CHAPTER XIII

MECHANICS

1. The Laws of Motion. The discovery of the Calculus opened the way to a clear insight into the rudimentary principles of physics, regarded as a quantitative science. In theory the simplest and, for the further developments, the most important branch of physics is Mechanics,* the laws of which find their expression and the appreciation of their meaning in the concepts and the methods of the Calculus.

Sir Isaac Newton (1642–1727), who with Leibniz founded the Calculus, formulated three laws governing the motion of a body. They are as follows:

FIRST LAW. *A body at rest remains at rest and a body in motion moves in a straight line with unchanging velocity, unless some external force acts on it.*

SECOND LAW. *The rate of change of the momentum of a body is proportional to the resultant external force that acts on the body.*

THIRD LAW. *Action and reaction are equal and opposite.*

By a *body* is here meant any rigid material distribution which is moving without rotation, as, for example, a freight car moving along a straight track, if the rotary motion of the wheels be neglected. It is often convenient to think of the mass as concentrated in a single point; *i.e.* to think of the body as replaced by a material particle, on which the same forces act. Moreover, as regards *force*, the conception is no more abstract than that of a *push* or a *pull*.

* It is customary in physics to take geometry for granted, as if it were a branch of mathematics. But in substance geometry is the noblest branch of physics.

The meaning of the First and the Third Law is obvious. In the Second Law the *momentum* of the body is to be understood as the product of its mass by its velocity, mv. And since, in the vast majority of cases which we meet in practice, the mass is constant, we have

$$\frac{d(mv)}{dt} = m\frac{dv}{dt}.$$

Now the rate at which the velocity changes, dv/dt, is what we commonly call *acceleration*, — we will denote it by α; — and hence the Second Law may be expressed as follows:

The mass times the acceleration is proportional to the force:

$$(1) \qquad m\alpha \propto f \qquad \text{or} \qquad m\alpha = \lambda f.$$

The factor λ is a physical constant. Its value depends on the units used. If these are the English units: foot, pound (mass), second, and pound (force), λ has the value 32. Hence equation (1) becomes:

$$(2) \qquad m\alpha = 32f.$$

Furthermore, since $v = \frac{ds}{dt}$, we have $\frac{dv}{dt} = \frac{d^2s}{dt^2}$. Thus

$$(3) \qquad \alpha = \frac{d^2s}{dt^2}.$$

Equation (2) now takes on the form:

$$(4) \qquad m\frac{d^2s}{dt^2} = 32f.$$

In applying the Second Law we are to regard a force which tends to increase s as positive, one that tends to decrease s as negative.

If forces oblique to the line of motion act on the body, each one must be broken up into a component along the line of motion and one perpendicular to this line. The latter component has no influence on the motion; the former component tends to produce motion. The force f of Newton's Second Law is obtained, when several forces act simultaneously, as the algebraic sum of all forces and components of forces along

the line of motion, taken positive when they tend to increase s, negative in the other case. We repeat that the body is thought of as moving without rotation and may, therefore, be conceived as a particle.

Finally, we will deduce a new expression for the acceleration. Starting with the obviously true equation

$$\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}$$

we replace ds/dt by its value, v. Hence

$$\alpha = v\frac{dv}{ds}. \tag{5}$$

The student should have clearly in mind these three forms for α:

$$\alpha=\frac{dv}{dt}=\frac{d^2 s}{dt^2}=v\frac{dv}{ds}. \tag{6}$$

Which one it is best to employ in a given case will become clear from the later examples.

Example 1. A freight train weighing 200 tons is drawn by a locomotive that exerts a draw-bar pull of 9 tons. 5 tons of this force are expended in overcoming frictional resistances. How much speed will the train have acquired at the end of a minute, if it starts from rest?

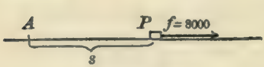


FIG. 113

Here we have

$$m = 200 \times 2000 = 400{,}000 \text{ lbs.},$$
$$f = 9 \times 2000 - 5 \times 2000 = 8000 \text{ lbs.*}$$

and hence equation (2) becomes

$$400{,}000\frac{dv}{dt} = 32 \times 8000,$$

or

$$\frac{dv}{dt}=\frac{16}{25}.$$

* The student must distinguish carefully between the two meanings of the word *pound*, namely (*a*) a *mass*, and (*b*) a *force*; — two totally different physical objects. Thus a pound of lead is a certain quantity of *matter*. If it is hung up by a string, the tension in the string is a pound of *force*.

Integrating with respect to t, we find:

$$v = \tfrac{16}{25}t + C.$$

Since $v = 0$ when $t = 0$, we must have $C = 0$, and hence

$$v = \tfrac{16}{25}t.$$

At the end of a minute, $t = 60$, and so

$$v = \tfrac{16}{25} \times 60 = 38.4 \text{ ft. per sec.}$$

To reduce feet per second to miles per hour it is convenient to notice that 30 miles an hour is equivalent to 44 ft. a second, as the student can readily verify; or roughly, 2 miles an hour corresponds to 3 ft. a second. Hence the speed in the present case is about two-thirds of 38.4, or 26 miles an hour.

Example 2. A stone is sent gliding over the ice with an initial velocity of 30 ft. a sec. If the coefficient of friction between the stone and the ice is $\frac{1}{10}$, how far will the stone go?

Here, the only force that we take account of is the retarding force of friction, and this amounts to one-tenth of a pound of force for every pound of mass there is in the stone. Hence, if there are m pounds of mass in the stone the force will be $\frac{1}{10}m$ lbs.,* and since it tends to decrease s, it is to be taken as negative:

FIG. 114

$$ma = 32\left(-\frac{m}{10}\right),$$

$$a = -\tfrac{16}{5}.$$

Now what we want is a relation between v and s, for the question is: How *far* ($s = ?$), when the stone *stops* ($v = 0$)? So we use the value (5) of a and thus obtain the equation:

$$v\frac{dv}{ds} = -\frac{16}{5},$$

or

$$vdv = -\tfrac{16}{5}ds.$$

* The student should notice that m is neither a mass nor a force, but a *number*, like all the other letters of Algebra, the Calculus, and Physics.

Hence $$\frac{v^2}{2} = -\frac{16}{5}s + C.$$

To determine C we have the data that, when $s = 0$, $v = 30$; hence, since in particular the equation must hold for these values,

$$\frac{30^2}{2} = 0 + C, \qquad C = 450,$$

and $$v^2 = 900 - \tfrac{32}{5}s.$$

When the stone stops, $v = 0$, and we have

$$0 = 900 - \tfrac{32}{5}s, \qquad s = 141 \text{ ft.}$$

EXERCISES*

1. If the train of Example 1 was moving at the rate of 4 m. an hour when we began to take notice, how fast would it be moving half a minute later? Give a complete solution, beginning with drawing the figure.

Ans. About 24 m. an h.; precisely, $24\frac{1}{15}$ m. an h.

2. A small boy sees a slide on the ice ahead, and runs for it. He reaches it with a speed of 8 miles an hour and slides 15 feet. How rough are his shoes, *i.e.* what is the coefficient of friction between his shoes and the ice? *Ans.* $\mu = .15$.

3. Show that, if the coefficient of friction between a sprinter's shoes and the track is $\frac{1}{12}$, his best possible record in a hundred-yard dash cannot be less than 15 seconds.

4. An electric car weighing 12 tons gets up a speed of 15 miles an hour in 10 seconds. Find the average force that acts on it, *i.e.* the constant force which would produce the same velocity in the same time.

* It is important that the student should work these exercises by the method set forth in the text, beginning each time by drawing a figure and marking (*i*) the *force*, by means of a directed right line, or vector, drawn preferably in red ink; and (*ii*) the *coordinate* used, as s or x, etc. He should not try to adapt such formulas of elementary physics as

$$v = at, \qquad s = \tfrac{1}{2}at^2, \qquad v^2 = 2as$$

to present purposes. For, the object of these simple exercises is to prepare the way for applications in which the force is not constant, and here the formulas just cited do not hold.

5. In the preceding problem, assume that the given speed is acquired after running 200 feet. Find the time required and the average force.

6. A train weighing 500 tons and running at the rate of 30 miles an hour is brought to rest by the brakes after running 600 feet. While it is being stopped it passes over a bridge. Find the force with which the bridge pulls on its anchorage.

Ans. 25.2 tons.

7. An electric car is starting on an icy track. The wheels skid and it takes the car 15 seconds to get up a speed of two miles an hour. Compute the coefficient of friction between the wheels and the track.

2. Absolute Units of Force. The units in terms of which we measure mass, space, time, and force are arbitrary. If we change one of them we thereby change the value of λ in Newton's Second Law, (1). Consequently, by changing the unit of force properly, the units of mass, space, and time being held fast, we can make $\lambda = 1$. Hence the

Definition. The absolute unit of force is that unit which makes $\lambda = 1$ in Newton's Second Law of Motion, (1):*

$$ma = f. \tag{7}$$

* We have already met a precisely similar question twice in the Calculus. In differentiating the function $\sin x$ we obtain the formula

$$D_x \sin x = \cos x$$

only when we measure angles in radians. Otherwise the formula reads

$$D_x \sin x = \lambda \cos x.$$

In particular, if the unit is a degree, $\lambda = \pi/180$. We may, therefore, define a radian as follows: The absolute unit of angle (the radian) is that unit which makes $\lambda = 1$ in the above equation.

Again, in differentiating the logarithm, we found

$$D_x \log_a x = (\log_a e)\frac{1}{x}.$$

This multiplier reduces to unity when we take $a = e$. Hence the definition: The absolute (natural) base of logarithms is that base which makes the multiplier $\log_a e$ in the above equation equal to unity.

In order to determine experimentally the absolute unit of force, we may allow a body to fall freely and observe how far it goes in a known time. Let the number g be the number of absolute units of force with which gravity attracts the unit of mass. Then the force, measured in absolute units, with which gravity attracts a body of m units of mass will be mg. Newton's Second Law (7) gives for this case:

$$m\frac{dv}{dt}=mg, \qquad \text{hence} \qquad \frac{dv}{dt}=g;$$

$$v=gt+C, \qquad C=0;$$

$$v=\frac{ds}{dt}=gt,$$

$$s=\tfrac{1}{2}gt^2+K, \qquad K=0,$$

and we have the law for freely falling bodies deduced directly from Newton's Second Law of Motion, the hypothesis being merely that the force of gravity is constant. Substituting in the last equation the observed values $s=S$, $t=T$, we get:

$$g=\frac{2S}{T^2}.$$

If we use English units for mass, space, and time, g has, to two significant figures, the value 32, *i.e.* the absolute unit of force in this system, a *poundal*, is equal nearly to *half an ounce.* If we use c.g.s. units, g ranges from 978 to 983 at different parts of the earth, and has in Cambridge the value 980. The absolute unit of force in this system is called the *dyne.*

Since g is equal to the acceleration with which a body falls freely under the attraction of gravity, g is called the *acceleration of gravity.* But this is not our definiton of g; it is a theorem about g that follows from Newton's Second Law of Motion.

The student can now readily prove the following theorem, which is often taken as the definition of the absolute unit of force in elementary physics: The absolute unit of force is that

force which, acting on the unit of mass for the unit of time, generates the unit of velocity.*

Incidentally we have obtained two of the equations for a freely falling body:

$$v = gt, \qquad s = \tfrac{1}{2}gt^2.$$

The third is found by setting $\alpha = v\,dv/ds$ and integrating:

$$v\frac{dv}{ds} = g,$$

$$\tfrac{1}{2}v^2 = gs + C; \qquad 0 = 0 + C,$$

$$v^2 = 2gs.$$

Example. A body is projected down an inclined plane with an initial velocity of v_0 feet per second. Determine the motion completely.

The forces which act are: the component of gravity, $mg \sin \gamma$

* Newton's Second Law can be written in the form:

$$\frac{f}{W} = \frac{\alpha}{g}, \qquad \text{or} \qquad \frac{W}{g}\alpha = f,$$

where W denotes the *weight* of the body, *i.e.* the *force* with which gravity attracts it; this force being measured in pounds or grammes or dynes or any other unit. This form of the Law has the advantage that it holds alike for gravitational and for absolute units of force, provided merely that f and W are both measured in terms of one and the same unit of force; and hence, in particular, the introduction of the absolute unit of force is rendered unnecessary. Moreover, this form is considered by some to be fool-proof with reference to the question of "when to put in and when to leave out g." It has the disadvantage of eliminating the conception of *mass* from mechanics. The fundamental units of physics are length, time, and mass; and it replaces them by length, time, and force. Thus the student fails to secure the training which will enable him to understand the standard treatises in physics and the classical treatment of mechanics by the masters of the science. He furthermore misses an opportunity to learn to dominate his units in physics, without which power he cannot hope to apply the results of physics contained in the literature of that science to the practical problems of engineering and the arts.

absolute units, down the plane, and the force of friction, $\mu R = \mu mg \cos\gamma$ up the plane. Hence

$$ma = mg \sin\gamma - \mu mg \cos\gamma$$

or

$$\frac{dv}{dt} = g \sin\gamma - \mu g \cos\gamma.$$

Integrating this equation, we get

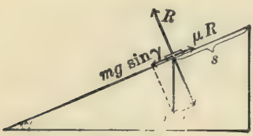


FIG. 115

$$v = g(\sin\gamma - \mu\cos\gamma)t + C,$$

$$v_0 = \quad 0 \quad + C,$$

(A) $$v = \frac{ds}{dt} = g(\sin\gamma - \mu\cos\gamma)t + v_0.$$

A second integration gives

(B) $$s = \tfrac{1}{2}g(\sin\gamma - \mu\cos\gamma)t^2 + v_0 t,$$

the constant of integration here being 0.

To find v in terms of s we may eliminate t between (A) and (B). Or we can begin by using formula (5) for the acceleration:

$$v\frac{dv}{ds} = g(\sin\gamma - \mu\cos\gamma),$$

$$\tfrac{1}{2}v^2 = g(\sin\gamma - \mu\cos\gamma)s \quad + K,$$

$$\tfrac{1}{2}v_0^2 = \quad 0 \quad + K,$$

$$v^2 = 2g(\sin\gamma - \mu\cos\gamma)s + v_0^2.$$

EXERCISES

1. If, in the example discussed in the text, the body is projected up the plane, find how far it will go up.

2. Determine the time it takes the body in Question 1 to reach the highest point.

3 Obtain the usual formulas for the motion of a body projected vertically:

$$v^2 = 2gs + v_0^2 \quad \text{or} \quad = -2gs + v_0^2;$$

$$v = gt + v_0 \quad \text{or} \quad = -gt + v_0;$$

$$s = \tfrac{1}{2}gt^2 + v_0 t \quad \text{or} \quad = -\tfrac{1}{2}gt^2 + v_0 t.$$

4. On the surface of the moon a pound weighs only one-sixth as much as on the surface of the earth. If a mouse can jump up 1 foot on the surface of the earth, how high could it jump on the surface of the moon? Compare the time it is in the air in the two cases, if the moon had an atmosphere.

5. A block of iron weighing 100 pounds rests on a smooth table. A cord, attached to the iron, runs over a smooth pulley at the edge of the table and carries a weight of 15 pounds, which hangs vertically. The system is released with the iron 10 feet from the pulley. How long will it be before the iron reaches the pulley, and how fast will it be moving?

Ans. 2.19 sec.; 9.1 ft. a sec.

6. Solve the same problem on the assumption that the table is rough, $\mu = \frac{1}{20}$, and that the pulley exerts a constant retarding force of 4 ounces.

7. Regarding the big locomotive exhibited at the World's Fair in 1905 by the Baltimore and Ohio Railroad the *Scientific American* said: "Previous to sending the engine to St. Louis, the engine was tested at Schenectady, where she took a 63-car train weighing 3,150 tons up a one-per-cent. grade."

Find how long it would take the engine to develop a speed of 15 m. per h. in the same train on the level, starting from rest, the draw-bar pull being assumed to be the same as on the grade.

8. If Sir Isaac Newton registered 170 pounds on a spring balance in an elevator at rest, and if, when the elevator was moving, he weighed only 169 pounds, what inference would he draw about the motion of the elevator?

9. What does a man whose weight is 180 pounds weigh in an elevator that is descending with an acceleration of 2 feet per second per second?

3. **Elastic Strings.** When an elastic string is stretched by a moderate amount, the tension, T, in the string is proportional to the stretching, *i.e.* to the difference, s, between the stretched and the unstretched length of the string:

$$(1) \qquad T \propto s, \qquad \text{or} \qquad T = ks,$$

where k is a physical constant, whose value depends both on the particular string and on the units employed.

Suppose, for example, that a string is stretched 6 in. by a force of 12 lbs.; to determine k. If we measure the force in gravitational units, *i.e.* pounds, then

$$T = 12 \qquad \text{when} \qquad s = \tfrac{1}{2}.$$

Hence, substituting these values in equation (1), we have:

$$12 = k\tfrac{1}{2}, \qquad \text{or} \qquad k = 24,$$

$$(2) \qquad T = 24s.$$

If we had chosen to measure the force in absolute units, *i.e.* poundals, then, since it takes (nearly) 32 of these units to make a pound, the given force of 12 pounds would be expressed as (nearly) 12×32, or precisely $12g$, poundals. Hence, substituting the present value of the force in (1), which, to avoid confusion, we will now write in the form:

$$T' = k's,$$

we have: $$12g = k'\tfrac{1}{2} \qquad \text{or} \qquad k' = 24g,$$

$$(3) \qquad T' = 24gs.$$

When the string is stretched 1 in., $s = \frac{1}{12}$, and the tension as given by (2) is $T = 2$, *i.e.* two pounds. Formula (3), on the other hand, gives $2g$, or 64 (nearly) as the value of the tension, expressed in terms of poundals, and this is right; for it takes 64 half-ounces to make 2 pounds, and so we should have $T' = 2g$.*

* It is easy to check an answer in any numerical case. The student has only to ask himself the question: "Have I expressed my force in pounds, or have I expressed it in terms of half-ounces?" Just as five dollars is expressed by the number 5 when we use the dollar as the unit, but by the number 500 when we use the cent, so, generally, the *smaller* the unit, the *larger* the number which expresses a given quantity.

The law of strings stated above is familiar to the student in the form of *Hooke's Law*:

$$T = E\frac{l' - l}{l},$$

where l is the natural, or unstretched, length of the string, and l', the stretched length; the coefficient E being Young's Modulus. For a given string, $E/l = k$ is constant, and $l' - l = s$ is variable.

EXERCISES

1. An elastic string is stretched 2 in. by a force of 3 lbs. Find the tension (*a*) in pounds; (*b*) in poundals, when it is stretched s ft. *Ans.* (*a*) $T = 18s$; (*b*) $T = 18gs$.

2. When the string of Question 1 is stretched 4 in., what is the tension (*a*) in terms of gravitational units; (*b*) in terms of absolute units? *Ans.* (*a*) 6 pounds; (*b*) 192 poundals.

3. An elastic string is stretched 1 cm. by a force of 100 grs. Find the tension (*a*) in grs.; (*b*) in dynes, when it is stretched s cm. *Ans.* (*a*) $100s$; (*b*) $98{,}000s$.

4. One end of an elastic string 3 ft. long is fastened to a peg at A, and a 2-pound weight is attached to the other end. The weight is gradually lowered till it is just supported by the string, and it is found that the length of the string has thus been doubled. Find the tension in the string when it is stretched s ft. *Ans.* $\frac{2}{3}s$ lbs.; $\frac{64}{3}s$ poundals.

4. A Problem of Motion. One end of the string considered in the text of § 3 is fastened to a peg at a point O of a smooth horizontal table; a weight of 3 lbs. is attached to the other end of the string and released from rest on the table with the string stretched one foot. How fast will the weight be moving when the string becomes slack?

The weight evidently describes a straight line from the starting point, A, toward the peg O, and we wish to know its velocity when it has reached a point B, one foot from A.

The solution is based on Newton's Second Law of Motion. It is convenient here to take as the coordinate, not the distance AP that the particle has travelled at any instant, but its distance s from B. The force which acts is the tension of the string; measured in absolute units it is $24gs$. Since it tends to decrease s, it is negative. Hence Newton's Law becomes:

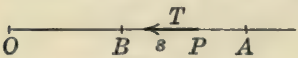


FIG. 116

$$(1) \qquad 3\frac{d^2s}{dt^2} = -24gs.$$

To integrate this equation, replace $\frac{d^2s}{dt^2}$ by its value $v\frac{dv}{ds}$:

$$(2) \qquad v\frac{dv}{ds} = -8gs.$$

Hence
$$vdv = -8gsds,$$

$$\int vdv = -8g\int sds,$$

$$(3) \qquad \frac{v^2}{2} = -4gs^2 + C.$$

To determine C, observe that initially, *i.e.* when the particle was released at A, $v = 0$ and $s = 1$. Hence

$$0 = -4g + C, \qquad C = 4g,$$

and (3) becomes

$$(4) \qquad v^2 = 8g(1 - s^2).$$

We have now determined the velocity of the particle at an arbitrary point of its path, and thus are in a position to find its velocity at the one point specified in the question proposed, namely, at B. Here, $s = 0$, and

$$v^2|_{s=0} = 8g = 8 \times 32, \qquad v|_{s=0} = 16 \text{ (ft. per sec.)}$$

EXERCISES*

1. The weight in the problem just discussed is projected from B along the table in the direction of OB produced with a

* In the following exercises and examples, it will be convenient to take the value of g as exactly 32 when English units are used.

velocity of 8 ft. per sec. Find how far it will go before it begins to return.

> *Ans.* Newton's equation is the same as before, and the integral, (3), is the same; but initially $s = 0$ and $v = 8$. Hence $C = 32$, and the answer is 6 inches.

2. If, in the example worked in the text, the table is rough and the coefficient of friction, μ, has the value $\frac{1}{4}$, how fast will the body be moving when it reaches B?

> *Ans.* Newton's equation now becomes:
>
> $$3\frac{d^2s}{dt^2} = -24gs + \tfrac{1}{4}\cdot 3g,$$
>
> and the answer is: $4\sqrt{15} = 15.49$ ft. per sec.

3. Solve the problem of Question 1, for a rough table, $\mu = \frac{1}{4}$.

> *Ans.* The required distance is the positive root of the equation $16s^2 + s - 4 = 0$, or $s = .4698$ ft., — about $5\frac{5}{8}$ in.

4. Find where the weight in Question 2 will come to rest if the string, after becoming slack, does not get in the way.

5. The 2 lb. weight of Question 4, § 3, is released from rest at a point B directly under the peg A and at a distance of 3 ft. from A; the string thus being taut, but not stretched. Find how far it will fall before it begins to rise. *Ans.* 6 ft.

6. If, in the last question, the weight is dropped from the peg at A, find how far it descends before it begins to rise.

> *Ans.* To a distance of $6 + 3\sqrt{3} = 11.196$ ft. below A.

7. If the weight in the last two questions is carried to a point 7 ft. below A and released, show that it will rise to a distance of 5 ft. below A before beginning to fall.

8. If, in the last question, the weight is released from a point 10 ft. below A, show that it will rise to a height of 1 ft. and 10 in. below A.

9. The string of the example studied in the text of § 3 is placed on a smooth inclined plane making an angle of 30° with the horizon, and one end is made fast to a peg at A in the

plane. If a weight of $1\frac{1}{8}$ lbs. be attached to the other end of the string and released from rest at A, find how far down the plane it will slide. Assume the unstretched length of the string to be 4 ft.

10. The same question if the plane is rough, $\mu = \frac{1}{6}\sqrt{3}$.

11. A cylindrical spar buoy (specific gravity $\frac{1}{2}$) is anchored so that it is just submerged at high water. If the cable should break at high tide, show that the spar would jump entirely out of the water.

Assume that the buoyancy of the water is always just equal to the weight of water displaced.

12. A particle of mass 2 lbs. lies on a rough horizontal table, and is fastened to a post by an elastic band whose unstretched length is 10 inches. The coefficient of friction is $\frac{1}{8}$, and the band is doubled in length by hanging it vertically with the weight at its lower end. If the particle be drawn out to a distance of 15 inches from the post and then projected directly away from the post with an initial velocity of 5 ft. a sec., find where it will stop for good.

5. Continuation; the Time. The time required by the body whose motion was studied in § 4 to reach the point B can be found as follows. From equation (4) we have:

$$v = \frac{ds}{dt} = \pm\sqrt{8g}\,\sqrt{1-s^2}. \tag{5}$$

Since s decreases as t increases, ds/dt is negative, and the lower sign holds. Replacing $\sqrt{8g}$ by its value, 16, we see that

$$\frac{ds}{dt} = -16\sqrt{1-s^2}. \tag{6}$$

This differential equation is readily solved by *separating the variables, i.e.* by transforming the equation so that only the variable s occurs on one side of the new equation, and only t on the other; thus

$$16\,dt = -\frac{ds}{\sqrt{1-s^2}}. \tag{7}$$

Hence $$16t = -\int \frac{ds}{\sqrt{1-s^2}} = -\sin^{-1}s + C.$$

If we measure the time from the instant when the body was released at A, then $t=0$ and $s=1$ are the initial values which determine C:

$$0 = -\sin^{-1}1 + C, \qquad C = \frac{\pi}{2}.$$

Thus $$16t = \frac{\pi}{2} - \sin^{-1}s.$$

The right-hand side of this equation has the value $\cos^{-1}s$, cf. p. 212, (8). Hence we have, as the final result,*

(8) $$16t = \cos^{-1}s, \qquad \text{or} \qquad s = \cos 16t.$$

This equation gives the time it takes the body to reach an arbitrary point of its path. In particular, the time from A to B is found by putting $s = 0$:

(9) $$16t = \cos^{-1}0 = \frac{\pi}{2}, \qquad t = \frac{\pi}{32} = .09818 \text{ sec.}$$

EXERCISES

1. Show that if the body, in the case just discussed, had been released from rest at any other distance from the peg, the string being stretched, the time to the point at which the string becomes slack would have been the same.

2. Show that, in any of these cases, it takes the body twice as long to cover the first half of its total path as it does to cover the remainder.

* In evaluating the above integral we might equally well have used the formula

$$\int \frac{ds}{\sqrt{1-s^2}} = -\cos^{-1}s + C'.$$

We should then have had:

$$16t = \cos^{-1}s - C'.$$

Substituting the initial values $t=0$, $s=1$ in this equation, we find:

$$0 = \cos^{-1}1 - C', \qquad \text{or} \qquad C' = 0,$$

and the final result is the same as before.

Find the time required to cover the entire path in the case of the following exercises at the close of § 4.

3. Exercise 1. *Ans.* $\frac{\pi}{32} = .09818.$

4. Exercise 5.

Ans. $t = \sqrt{\frac{3}{32}} \int \frac{ds}{\sqrt{6s - s^2}}$; total time, $\pi\sqrt{\frac{3}{32}} = .9618$ sec.

5. Exercise 6. *Ans.* $t = \sqrt{\frac{3}{g}}\left[\frac{\pi}{2} + \sin^{-1}\frac{1}{\sqrt{3}}\right] = ?$

6. Exercise 7. *Ans.* .9618 sec.

7. Exercise 9. **8.** Exercise 10. **9.** Exercise 8.

6. Simple Harmonic Motion. The simplest and most important case of oscillatory motion which occurs in nature is that known as *Simple Harmonic Motion.* It is illustrated with the least amount of technical detail by the following example, or by the first Exercise below.

Example. A hole is bored through the centre of the earth, a stone is inserted, the air is exhausted, and the stone is released from rest at the surface of the earth. To determine the motion.

The earth is here considered as a homogeneous sphere, at rest in space. Its attraction, F, on the stone diminishes as the stone nears the centre, and it can be shown to be proportional, at any point of the hole, to the distance of the stone from the centre:

$$F \propto r, \qquad \text{or} \qquad F = kr.$$

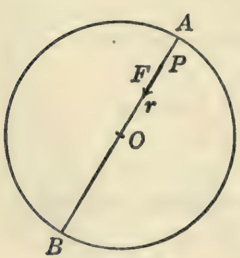


FIG. 117

To determine the constant k, observe that, at the surface, $r = R$ (the radius of the earth), and, if we measure F in absolute units, $F = mg$, where m denotes the mass of the stone. Hence

$$mg = kR, \qquad \text{or} \qquad k = \frac{mg}{R},$$

and

$$F = \frac{mg}{R} r.$$

As the coordinate of the stone we will take its distance, r, from the centre of the earth. Then Newton's Second Law gives us:

$$m\frac{d^2r}{dt^2}=-\frac{mg}{R}r. \tag{1}$$

For, when r is positive, the force tends to decrease r, and so is negative. When r is negative, the force tends to increase r algebraically, and so is positive. Hence (1) is right in all cases.

In order to integrate equation (1), which can be written in the form

$$\frac{d^2r}{dt^2}=-\frac{g}{R}r, \tag{2}$$

we employ the device of multiplying through by $2\,dr/dt$:

$$2\frac{dr}{dt}\frac{d^2r}{dt^2}=-\frac{2g}{R}r\frac{dr}{dt}.$$

The left-hand side thus becomes $\dfrac{d}{dt}\left(\dfrac{dr}{dt}\right)^2$. Hence each side can be integrated with respect to t:*

$$\int\frac{d}{dt}\left(\frac{dr}{dt}\right)^2dt=-\frac{2g}{R}\int r\frac{dr}{dt}dt,$$

or

$$\left(\frac{dr}{dt}\right)^2=-\frac{2g}{R}\int r\,dr=-\frac{g}{R}\cdot r^2+C.$$

* This method can be applied to any differential equation of the form

$$\frac{d^2y}{dx^2}=f(y).$$

Multiply through by $2\,dy/dx$:

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}=2f(y)\frac{dy}{dx}.$$

The left-hand side thus becomes $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)^2$. Hence

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2=2f(y)\frac{dy}{dx}, \qquad \text{or} \qquad d\left(\frac{dy}{dx}\right)^2=2f(y)\,dy.$$

Integrating, we have

$$\left(\frac{dy}{dx}\right)^2=2\int f(y)\,dy.$$

To determine C, observe that initially, *i.e.* when the stone was at A, $r = R$ and the velocity, $dr/dt, = 0$. Hence

$$0 = -\frac{g}{R}R^2 + C, \qquad \text{or} \qquad C = \frac{g}{R}R^2.$$

Thus finally:

$$\left(\frac{dr}{dt}\right)^2 = \frac{g}{R}(R^2 - r^2). \tag{3}$$

At the centre of the earth, $r = 0$, and $(dr/dt)^2 = gR$. If we take the radius of the earth as 4000 miles, then $R = 4000 \times 5280$, $g = 32$, and the velocity is about 26,000 ft. a sec., or approximately 5 miles a second.

The stone keeps on with diminishing speed and comes to rest for an instant when $r = -R$, *i.e.* it just reaches the other side of the earth, and then falls back. Thus it oscillates throughout the whole length of the hole, reaching the surface at the end of each excursion, and continuing this motion forever. The result is not unreasonable, for there is no damping of any sort, — no friction or air resistance.

The Time. To find the time we proceed as in § 5. From equation (3) it follows that

$$\frac{dr}{dt} = -\sqrt{\frac{g}{R}}\sqrt{R^2 - r^2}.$$

Hence, separating the variables, we have:

$$dt = -\sqrt{\frac{R}{g}}\frac{dr}{\sqrt{R^2 - r^2}}, \qquad t = -\sqrt{\frac{R}{g}}\int\frac{dr}{\sqrt{R^2 - r^2}},$$

or

$$t = \sqrt{\frac{R}{g}}\cos^{-1}\frac{r}{R} + C.$$

Initially, $t = 0$ and $r = R$; thus $C = 0$, and

$$t = \sqrt{\frac{R}{g}}\cos^{-1}\frac{r}{R}, \qquad \text{or} \qquad r = R\cos\left(t\sqrt{\frac{g}{R}}\right). \tag{4}$$

The time from A to O is found by putting $r = 0$:

$$t\Big|_{r=0} = \frac{\pi}{2}\sqrt{\frac{R}{g}}.$$

On computing the value of this expression it is seen to be 21 min. and 16 sec. The time from A to B is twice the above. Hence the time of a complete excursion, from A to B and back to A is

$$2\pi\sqrt{\frac{R}{g}}.$$

This time is known as the *period* of the oscillation.*

The General Case. Simple Harmonic Motion is always dominated by the differential equation

$$\frac{d^2x}{dt^2} = -n^2x, \tag{A}$$

where the coordinate x characterizes the displacement from the position of no force. This equation can be integrated as in the special case above, and it is found that

$$\left(\frac{dx}{dt}\right)^2 = n^2(h^2 - x^2), \tag{B}$$

where h denotes the value of x which corresponds to the extreme displacement. The velocity when $x = 0$ is numerically nh, and thus is proportional both to n and to h. A second integration gives

$$x = h\cos nt, \tag{C}$$

provided the time is measured from an instant when $x = h$. The period, T, is inversely proportional to n:

$$T = \frac{2\pi}{n}, \tag{D}$$

and the amplitude is $2h$. Thus the period is independent of the amplitude.

EXERCISES

1. Two strings like the one described in the text of § 3 are fastened, one end of each, to two pegs, A and B, on a smooth horizontal table, the distance AB being double the length of

* In the first equation (4) the principal value of the anti-cosine holds during the first passage of the stone from A to B. The second equation (4) holds without restriction.

either string, and the other end of each string is made fast to a 3 lb. weight, which is placed at O, the mid-point of AB Thus each string is taut, but not stretched. The weight being moved to a point C between O and A and then released from rest, show that it oscillates with simple harmonic motion. Find the velocity with which it passes O and the period of the oscillation. It is assumed that the string which is slack in no wise interferes with or influences the motion.

Ans. The differential equation which dominates the motion is $\frac{d^2x}{dt^2} = -256x$, where x denotes the displacement of the 3 lb. weight; hence the motion is simple harmonic motion. The required velocity is numerically $16h$, where h denotes the maximum displacement. The period is $2\pi/16 = .3927$ sec.

2. Work the same problem for two strings like the one of question 4, § 3, and a 2 lb. weight.

3. Show that the motion of example 7, § 4, is simple harmonic motion, and find the period.

4. If a straight hole were bored through the earth from Boston to London, a smooth tube containing a letter inserted, the air exhausted from the tube, and the letter released at Boston, how long would it take the letter to reach London?

5. If in the problem of Question 9, § 4, the weight were released with the string taut, but not stretched, and directed straight down the plane, show that the weight would execute simple harmonic motion. Determine the amplitude and the period.

6. Work the problem of the text for the moon; cf. the data in § 7.

7. A steel wire of one square millimeter cross-section is hung up in Bunker Hill Monument, and a weight of 25 kilogrammes is fastened to its lower end and carefully brought to rest. The weight is then given a slight vertical displacement. Determine the period of the oscillation.

Given that the force required to double the length of the wire is 21,000 kilogrammes, and that the length of the wire is 210 feet. *Ans.* A little over half a second.

8. A number of iron weights are attached to one end of a long round wooden spar, so that, when left to itself, the spar floats vertically in water. A ten-kilogramme weight having become accidentally detached, the spar is seen to oscillate with a period of 4 seconds. The radius of the spar is 10 centimetres. Find the sum of the weights of the spar and attached iron. Through what distance does the spar oscillate?

Ans. (*a*) About 125 kilogrammes; (*b*) 0.64 metre.

7. Motion under the Attraction of Gravitation. *Problem.* To find the velocity which a stone acquires in falling to the earth from interstellar space.

Assume the earth to be at rest and consider only the force which the earth exerts. Let the stone be released from rest at A, and let r be its distance from the centre O of the earth at any subsequent instant. Then the force, F, acting on it is, by the law of gravitation, Chapter XII, § 18, inversely proportional to r:

$$F = \frac{\lambda}{r^2}.$$

Fig. 118

Since $F = mg$ when $r = R$, the radius of the earth,

$$mg = \frac{\lambda}{R^2} \qquad \text{and} \qquad F = \frac{mgR^2}{r^2}.$$

Newton's Second Law of Motion here takes on the form:

$$m\frac{d^2 r}{dt^2} = -\frac{mgR^2}{r^2}.$$

Hence

(1) $$\frac{d^2 r}{dt^2} = -\frac{gR^2}{r^2}.$$

To integrate this equation, we employ the method of § 6 and multiply by $2\,dr/dt$:

$$2\frac{dr}{dt}\frac{d^2r}{dt^2} = -\frac{2gR^2}{r^2}\frac{dr}{dt}, \qquad \text{or} \qquad \frac{d}{dt}\left(\frac{dr}{dt}\right)^2 = -\frac{2gR^2}{r^2}\frac{dr}{dt}.$$

Integrating with respect to t we get:

$$\frac{dr^2}{dt^2} = -2gR^2\int\frac{dr}{r^2} = \frac{2gR^2}{r} + C.$$

Initially $dr/dt = 0$ and $r = l$; hence

$$0 = \frac{2gR^2}{l} + C, \qquad C = -\frac{2gR^2}{l}.$$

(2) $$\frac{dr^2}{dt^2} = 2gR^2\left(\frac{1}{r} - \frac{1}{l}\right).$$

Since dr/dt is numerically equal to the velocity, the velocity V at the surface of the earth is given by the equation:

$$V^2 = 2gR^2\left(\frac{1}{R} - \frac{1}{l}\right).$$

If l is very great, the last term in the parenthesis is small, and so, no matter how great l is, V can never quite equal $\sqrt{2gR}$. Here $g = 32$, $R = 4000 \times 5280$, and hence the velocity in question is about 36,000 feet, or 7 miles, a second.

This solution neglects the retarding effect of the atmosphere; but as the atmosphere is very rare at a height of 50 miles from the earth's surface, the result is reliable down to a point comparatively near the earth.

In order to find the time it would take the stone to fall, write (2) in the form

$$\frac{dr}{dt} = -\sqrt{2gR^2}\sqrt{\frac{l-r}{lr}}.$$

Hence $$dt = -\frac{\sqrt{l}}{8R}\frac{r\,dr}{\sqrt{lr - r^2}}$$

and $$t = -\frac{\sqrt{l}}{8R}\int\frac{r\,dr}{\sqrt{lr - r^2}}.$$

Turning to Peirce's *Tables*, No. 169, we find

$$\int \frac{r\,dr}{\sqrt{lr-r^2}} = -\sqrt{lr-r^2} + \frac{l}{2}\int \frac{dr}{\sqrt{lr-r^2}}$$

$$= -\sqrt{lr-r^2} + \frac{l}{2}\sin^{-1}\frac{2r-l}{l}.$$

Thus
$$t = \frac{\sqrt{l}}{8R}\left\{\sqrt{lr-r^2} - \frac{l}{2}\sin^{-1}\frac{2r-l}{l}\right\} + K.$$

Initially $t=0$ and $r=l$:

$$0 = \frac{\sqrt{l}}{8R}\left\{0 - \frac{l}{2}\frac{\pi}{2}\right\} + K.$$

Finally, then,

$$t = \frac{\sqrt{l}}{8R}\left\{\sqrt{lr-r^2} + \frac{l}{2}\left[\frac{\pi}{2} - \sin^{-1}\frac{2r-l}{l}\right]\right\}.$$

For purposes of computation, a better form of this equation is the following:

(3)
$$t = \frac{l^{\frac{3}{2}}}{8R}\left\{\sqrt{\frac{r}{l} - \left(\frac{r}{l}\right)^2} + \frac{1}{2}\cos^{-1}\left(2\frac{r}{l} - 1\right)\right\}.$$

EXERCISES*

1. If the earth had no atmosphere, with what velocity would a stone have to be projected from the earth's surface, in order not to come back?

* In working these exercises, the following data may be used:

Radius of the moon, $\frac{3}{11}$ that of the earth.

Mass of moon, $\frac{1}{81}$ that of earth.

Mean distance of moon from earth, 237,000 miles.

Acceleration of gravity on the surface of the moon, $\frac{1}{6}$ that on the surface of the earth.

Diameter of sun, 860,000 miles.

Mass of sun, 333,000 that of the earth.

Mean distance of earth from sun, 93,000,000 miles.

Acceleration of gravity on the surface of the sun, 905 ft. per sec. per sec.

2. If the moon were stopped in its course, how long would it take it to fall to the earth? Regard the earth as stationary

Ans. 4 days, 18 hrs., 10 min.

3. Solve the preceding problem accurately, assuming that the earth and the moon are released from rest in interstellar space at their present mean distance apart. Their common centre of gravity will then remain stationary.

4. The same problem for the earth and the sun.

5. If the earth and the moon were held at rest at their present mean distance apart, with what velocity would a projectile have to be shot from the surface of the moon, in order to reach the earth?

6. If the earth and the moon were held at rest at their present mean distance apart, and a stone were placed between them at the point of no force and then slightly displaced toward the earth, with what velocity would it reach the earth?

7. If a hole were bored through the centre of the moon, assumed spherical, homogeneous, and at rest in interstellar space, and a stone dropped in, how long would it take the stone to reach the other side?

8. Show that if two spheres, each one foot in diameter and of density equal to the earth's mean density (specific gravity, 5.6) were placed with their surfaces $\frac{1}{4}$ of an inch apart and were acted on by no other forces than their mutual attractions, they would come together in about five minutes and a half. Given that the spheres attract as if all their mass were concentrated at their centres.

8. Constrained Motion. If a particle is constrained to describe a given path, as in the case, for example, of a simple pendulum, then the form which Newton's Second Law of Motion assumes is that the product of the mass by the acceleration along the path is equal to the component, along the path, of the resultant of all the forces that act.

Consider the simple pendulum. Here

$$m\frac{d^2s}{dt^2} = -mg\sin\theta,$$

and since $s = l\theta$,

$$\text{(1)} \qquad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

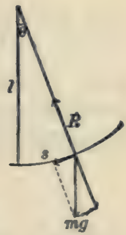


FIG. 119

This differential equation is characteristic for *Simple Pendulum Motion.* We can obtain a first integral by the method of § 6:

$$2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = -\frac{2g}{l}\sin\theta\frac{d\theta}{dt},$$

$$\frac{d\theta^2}{dt^2} = -\frac{2g}{l}\int\sin\theta\,d\theta = \frac{2g}{l}\cos\theta + C,$$

$$0 = \frac{2g}{l}\cos\alpha + C,$$

where α is the initial angle; hence

$$\text{(2)} \qquad \frac{d\theta^2}{dt^2} = \frac{2g}{l}(\cos\theta - \cos\alpha).$$

The velocity in the path at the lowest point is l times the angular velocity for $\theta = 0$, or $\sqrt{2gl(1-\cos\alpha)}$, and is the same that would have been acquired if the bob had fallen freely under the force of gravity through the same difference in level.

If we attempt to obtain the time by integrating equation (2), we are led to the equation:

$$t = \sqrt{\frac{l}{2g}}\int\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

This integral cannot be expressed in terms of the functions at present at our disposal. It is an Elliptic Integral. When θ, however, is small, $\sin\theta$ differs from θ by only a small percentage of either quantity, Chap. V, § 3, and hence we may expect to obtain a good approximation to the actual motion if we replace $\sin\theta$ in (1) by θ:

$$\text{(3)} \qquad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta.$$

This latter equation is of the type of the differential equation of Simple Harmonic Motion, § 6, (A), n^2 having here the value g/l. Hence, when a simple pendulum swings through a small amplitude, its motion is approximately harmonic and its period is approximately

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

The Tautochrone. A question that interested the mathematicians of the eighteenth century was this: In what curve should a pendulum swing in order that the period of oscillation may be absolutely independent of the amplitude? It turns out that the cycloid has this property. For, the differential equation of motion is

$$m\frac{d^2s}{dt^2} = -mg\sin\tau,$$

where s is measured from the lowest point, and since, from Ex. 8, p. 275,

$$s = 4a\sin\tau,$$

we have
$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s.$$

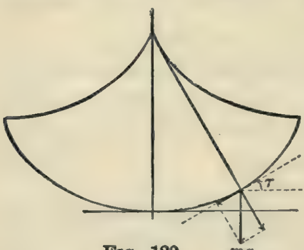


Fig. 120

This is the differential equation of Simple Harmonic Motion, § 6, (A), and hence the period of the oscillation,

$$T = 2\pi\sqrt{\frac{4a}{g}} = 4\pi\sqrt{\frac{a}{g}},$$

is independent of the amplitude.

A cycloidal pendulum may be constructed by causing the cord of the pendulum to wind on the evolute of the path. The resistances due to the stiffness of the cord as it winds up and unwinds would thus be slight; but in time they would become appreciable.

Smooth Bead on an Arbitrary Wire. Suppose a bead slides on a smooth wire of any shape whatever. We proceed to show that its velocity at any point will be the same as what

the bead would have acquired in falling freely under the force of gravity the same difference in level.

Special cases of this theorem have already presented themselves in the inclined plane and the simple pendulum. We shall restrict ourselves to plane curves, but the proof can be extended without difficulty to twisted curves.

Newton's Second Law of Motion gives

$$m\frac{d^2s}{dt^2} = mg\cos\tau = mg\frac{dx}{ds}.$$

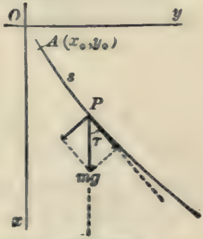


FIG. 121

Hence $$2\frac{ds}{dt}\frac{d^2s}{dt^2} = 2g\frac{dx}{ds}\frac{ds}{dt} = 2g\frac{dx}{dt}.$$

Integrating this equation with respect to t, we find:

$$\frac{ds^2}{dt^2} = 2gx + C.$$

If we suppose the bead to start from rest at A, then

$$0 = 2gx_0 + C,$$

(𝔄) $$v^2 = \frac{ds^2}{dt^2} = 2g(x - x_0).$$

But the velocity that a body falling freely a distance of $x - x_0$ attains is expressed by precisely the same formula, and thus the theorem is established.

In the more general case that the bead passes the point A with a velocity v_0 we have:

$$v_0^2 = 2gx_0 + C,$$

(𝔄′) $$v^2 - v_0^2 = 2g(x - x_0).$$

Thus it is seen that the velocity at P is the same that the bead would have acquired at the second level if it had been projected vertically from the first with velocity v_0.

The theorem also asserts that the sum of the kinetic and potential energies of the bead is constant, or that the change in kinetic energy is equal to the work done on the bead; cf. § 9.

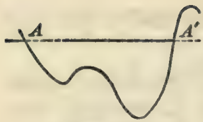


FIG. 122

If the bead starts from rest at A, it will continue to slide till it reaches the end of the wire or comes to a point A' at the same level as A. In the latter case it will in general just rise to the point A' and then retrace its path back to A. But if the tangent to the curve at A' is horizontal, the bead may approach A' as a limiting position without ever reaching it.

EXERCISES

1. A bead slides on a smooth vertical circle. It is projected from the lowest point with a velocity equal to that which it would acquire in falling from rest from the highest point. Show that it will approach the highest point as a limit which it will never reach.

2. From the general theorem ($\mathfrak{A}$) deduce the first integral (2) of the differential equation (1).

9. **Kinetic Energy and Work.** Let a particle, of mass m, describe a rectilinear path under the action of a force, F, directed along the path and varying continuously. Let x be the coordinate of the particle, and let F be positive when it tends to increase x; negative, in the other case. Then F is a continuous function of x: $F=f(x)$. Newton's Second Law becomes:

$$m\frac{d^2x}{dt^2}=f(x).$$

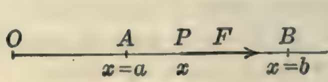


FIG. 123

Multiplying through by dx/dt, we have:

$$m\frac{dx}{dt}\frac{d^2x}{dt^2}=f(x)\frac{dx}{dt}, \qquad \text{or} \qquad \frac{d}{dt}\left[\frac{m}{2}\frac{dx^2}{dt^2}\right]=f(x)\frac{dx}{dt}.$$

If we integrate each side of this last equation between the limits $t=t_0$ (when $x=a$) and $t=t_1$ (when $x=b$) we have:

$$\int_{t_0}^{t_1}\frac{d}{dt}\left[\frac{m}{2}\frac{dx^2}{dt^2}\right]dt=\int_{t_0}^{t_1}f(x)\frac{dx}{dt}dt=\int_a^b f(x)\,dx.$$

This last integral represents precisely the work, W, done on the particle by the force F, Chap. XII, § 20. On the other hand, the indefinite integral

$$\int \frac{d}{dt}\left[\frac{m}{2}\frac{dx^2}{dt^2}\right] dt = \frac{m}{2}\frac{dx^2}{dt^2} = \frac{mv^2}{2},$$

where v denotes the velocity of m. When this integral is taken between the limits $t = t_0$ $(x = a)$ and $t = t_1$ $(x = b)$, we have:

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$

Hence
$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = W,$$

or *the change in the kinetic energy of the particle m is equal to the work done by the force F which acts on it.*

EXERCISES

By means of the theorem just established, obtain a first integral of the differential equation arising from Newton's Second Law of Motion in each of the following cases:

1. The problem worked in the text of § 4.

2. Exercise 1, § 4. 3. Exercise 2, § 4. 4. Exercise 4, § 4.

5. The problem worked in the text of § 7.

10. Motion in a Resisting Medium. When a body moves through the air or through the water, these media oppose resistance, the magnitude of which depends on the velocity, but does not follow any simple mathematical law. For low velocities up to 5 or 10 miles per hour, the resistance R can be expressed approximately by the formula:

(1) $$R = av,$$

where a is a constant depending both on the medium and on the size and shape of the body, but not on its mass. For higher velocities up to the velocity of sound (1082 ft. a sec.) the formula

(2) $$R = cv^2$$

gives a sufficient approximation for many of the cases that arise in practice. We shall speak of other formulas in the next paragraph.

Problem 1. A man is rowing in still water at the rate of 3 miles an hour, when he ships his oars. Determine the subsequent motion of the boat.

Here Newton's Second Law gives us:

$$m\frac{dv}{dt} = -av. \tag{3}$$

Hence

$$dt = -\frac{m}{a}\frac{dv}{v},$$

$$t = \frac{m}{a}\log\frac{v_0}{v}, \tag{4}$$

where v_0 is the initial velocity, nearly $4\frac{1}{2}$ ft. a sec.

To solve (4) for v, observe that

$$\frac{at}{m} = \log\frac{v_0}{v}, \qquad \text{or} \qquad e^{\frac{at}{m}} = \frac{v_0}{v}.$$

Hence

$$v = v_0 e^{-\frac{at}{m}}. \tag{5}$$

It might appear from (5) that the boat would never come to rest, but would move more and more slowly, since

$$\lim_{t=\infty} e^{-\frac{at}{m}} = 0.$$

We warn the student, however, against such a conclusion. For the approximation we are using, $R = av$, holds only for a limited time, and even for that time is at best an *approximation.* It will probably not be many minutes before the boat is drifting sidewise, and the value of a for this aspect of the boat would be quite different, — if indeed the approximation $R = av$ could be used at all.

To determine the distance travelled, we have from (3):

$$mv\frac{dv}{ds} = -av,$$

and consequently:

(6) $$v = v_0 - \frac{a}{m}s.$$

Hence, even if the above law of resistance held up to the limit, the boat would not travel an infinite distance, but would approach a point distant

$$S = \frac{mv_0}{a}$$

feet from the starting point, the distance traversed thus being proportional to the initial momentum.

Finally, to get a relation between s and t, integrate (5):

$$\frac{ds}{dt} = v_0 e^{-\frac{at}{m}},$$

(7) $$s = \frac{mv_0}{a}(1 - e^{-\frac{at}{m}}).$$

From this result is also evident that the boat will never cover a distance of S ft. while the above approximation lasts.

EXERCISE

If the man and the boat together weigh 300 lbs. and if a steady force of 3 lbs. is just sufficient to maintain a speed of 3 miles an hour in still water, show that when the boat has gone 20 ft., the speed has fallen off by a little less than a mile an hour.

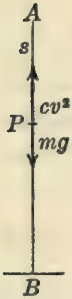
Problem 2. A drop of rain falls from a cloud with an initial velocity of v_0 ft. a sec. Determine the motion.

We assume that the drop is already of its final size, — not gathering further moisture as it proceeds, — and take as the law of resistance:

$$R = cv^2.$$

The forces which act are (i) the force of gravity, mg, downward, and (ii) the resistance of the air, cv^2, upward. As the coordinate of the particle we will take the distance AP, Fig. 124, which it has fallen. Then, Newton's Second Law becomes:

$$m\frac{dv}{dt} = mg - cv^2.$$



Hence
$$v\frac{dv}{ds} = \frac{mg - cv^2}{m},$$

$$ds = \frac{mv\,dv}{mg - cv^2},$$

$$s = -\frac{m}{2c}\log(mg - cv^2) + C,$$

$$0 = -\frac{m}{2c}\log(mg - cv_0^2) + C,$$

and thus finally

FIG. 124

(8)
$$s = \frac{m}{2c}\log\frac{mg - cv_0^2}{mg - cv^2}.$$

Solving for v we have

$$e^{\frac{2cs}{m}} = \frac{mg - cv_0^2}{mg - cv^2},$$

(9)
$$v^2 = \frac{mg}{c} - \frac{mg - cv_0^2}{c}e^{-\frac{2cs}{m}}.$$

When s increases indefinitely, the last term approaches 0 as its limit, and hence the velocity v can never exceed (or quite equal) $\bar{v} = \sqrt{mg/c}$ ft. a sec. This is known as the *limiting velocity*. It is independent of the height and also of the initial velocity, and is practically attained by the rain as it falls, for a rain drop is not moving sensibly faster when it reaches the ground than it was at the top of a high building.

EXERCISES

1. Work Problem 2, taking as the coordinate of the rain drop its height above the ground.

2. Find the time in terms of the velocity and the velocity in terms of the time in Problem 2.

3. Show that, if a charge of shot be fired vertically upward, it will return with a velocity about $3\frac{1}{3}$ times that of rain drops of the same size; and that if it be fired directly downward from a balloon two miles high, the velocity will not be appreciably greater.

4. Determine the height to which the shot will rise in Ex. 3, and show that the time to the highest point is

$$t = \sqrt{\frac{m}{gc}} \tan^{-1}\left(v_0 \sqrt{\frac{c}{mg}}\right),$$

where v_0 is the initial velocity.

11. Graph of the Resistance. The resistance which the atmosphere or water opposes to a body of a given size and shape can in many cases be determined experimentally with a reasonable degree of precision and thus the graph of the resistance:

$$R = f(v)$$

can be plotted. The mathematical problem then presents itself of representing the curve with sufficient accuracy by means of a simple function of v. In the problem of vertical motion in the atmosphere, Problem 2, § 10,

$$m\frac{dv}{dt} = mg \pm f(v),$$

FIG. 125

according as the body is going up or coming down, s being measured positively downward. Now if we approximate to $f(v)$ by means of a quadratic polynomial or a fractional linear function,

$$a + bv + cv^2 \qquad \text{or} \qquad \frac{\alpha + \beta v}{\gamma + \delta v},$$

we can integrate the resulting equation readily. And it is obvious that we can so approximate, — at least, for a restricted range of values for v.

Another case of interest is that in which the resistance of the medium is the only force that acts, as in Problem 1:

$$m\frac{dv}{dt} = -f(v).$$

A convenient approximation for the purposes of integration is

$$f(v) = av^b.$$

Here a and b are merely arbitrary constants, enabling us to impose two arbitrary conditions on the curve, — for example,

to make it go through two given points, — and are to be determined so as to yield a good approximation to the physical law. Sometimes the simple values $b = 1$, 2, 3 can be used with advantage. But we must not confuse these approximate formulas with similarly appearing formulas that represent exact physical laws. Thus, in geometry, the areas of similar surfaces and the volumes of similar solids are proportional to the squares or cubes of corresponding linear dimensions. This law expresses a fact that holds to the finest degree of accuracy of which physical measurements have shown themselves to be capable and with no restriction whatever on the size of the bodies. But the law $R = av^2$ or $R = cv^3$ ceases to hold, *i.e.* to interpret nature within the limits of precision of physical measurements, when v transcends certain restricted limits, and the student must be careful to bear this fact in mind.

EXERCISES

Work out the relations between v and s, and those between v and t, if the only force acting is the resistance of the medium, which is represented by the formula:

1. $R = a + bv + cv^2$. 2. $R = \dfrac{\alpha + \beta v}{\gamma + \delta v}$. 3. $R = a v^b$.

4. Show that it would be feasible mathematically to use the formulas of Questions 1 and 2 in the case of the falling rain drop.

5. A train weighing 300 tons, inclusive of the locomotive, can just be kept in motion on a level track by a force of 3 pounds to the ton. The locomotive is able to maintain a speed of 60 miles an hour, the horse power developed being reckoned as 1300. Assuming that the frictional resistances are the same at high speeds as at low ones and that the resistance of the air is proportional to the square of the velocity, find by how much the speed of the train will have dropped off in running half a mile if the steam is cut off with the train at full speed.

6. A man and a parachute weigh 150 pounds. How large must the parachute be that the man may trust himself to it at any height, if 25 ft. a sec. is a safe velocity with which to reach the ground? Given that the resistance of the air is as the square of the velocity and is equal to 2 pounds per square foot of opposing surface for a velocity of 30 ft. a sec.

Ans. About 12 ft. in diameter.

7. A toboggan slide of constant slope is a quarter of a mile long and has a fall of 200 ft. Assuming that the coefficient of friction is $\frac{3}{100}$, that the resistance of the air is proportional to the square of the velocity and is equal to 2 pounds per square foot of opposing surface for a velocity of 30 ft. a sec., and that a loaded toboggan weighs 300 pounds and presents a surface of 3 sq. ft. to the resistance of the air; find the velocity acquired during the descent and the time required to reach the bottom.

Find the limit of velocity that could be acquired by a toboggan under the given conditions if the hill were of infinite length.

Ans. (*a*) 68 ft. a sec.; (*b*) 30 secs.; (*c*) 74 ft. a sec.

8. The ropes of an elevator break and the elevator falls without obstruction till it enters an air chamber at the bottom of the shaft. The elevator weighs 2 tons and it falls from a height of 50 ft. The cross-section of the well is 6×6 ft. and its depth is 12 ft. If no air escaped from the well, how far would the elevator sink in? What would be the maximum weight of a man of 170 pounds? Given that the pressure and the volume of air when compressed without gain or loss of heat follow the law:

$$pv^{1.41} = \text{const.},$$

and that the atmospheric pressure is 14 pounds to the square inch.

12. Motion of a Projectile. *Problem.* To find the path of a projectile acted on only by the force of gravity.

The degree of accuracy of the approximation to the true motion obtained in the following solution depends on the

projectile and on the velocity with which it moves. For a cannon ball it is crude, though suggestive, whereas for the 16 lb. shot used in putting the shot it is decidedly good.

Hitherto we have known the path of the body; here we do not. We may state Newton's Second Law of Motion for a plane path as follows:*

$$(1) \qquad \begin{cases} m\dfrac{d^2x}{dt^2} = X, \\[2ex] m\dfrac{d^2y}{dt^2} = Y, \end{cases}$$

where X, Y are the components of the resultant force along the axes, measured in absolute units.

In the present case $X = 0$, $Y = -mg$, and we have

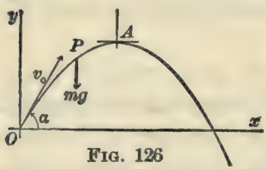


FIG. 126

$$(2) \qquad \begin{cases} m\dfrac{d^2x}{dt^2} = 0, \\[2ex] m\dfrac{d^2y}{dt^2} = -mg. \end{cases}$$

If we suppose the body projected from O with velocity v_0 at an angle α with the horizontal, the integration of these equations gives:

$$\frac{dx}{dt} = C = v_0 \cos\alpha, \qquad x = v_0 t \cos\alpha;$$

$$\frac{dy}{dt} = v_0 \sin\alpha - gt, \qquad y = v_0 t \sin\alpha - \tfrac{1}{2}gt^2.$$

Eliminating t we get:

$$(3) \qquad y = x \tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}.$$

The curve has a maximum at the point A: (x_1, y_1),

$$x_1 = \frac{v_0^2 \sin\alpha\cos\alpha}{g}. \qquad y_1 = \frac{v_0^2\sin^2\alpha}{2g}.$$

* The form of Newton's Second Law that covers all cases, both in the plane and in space, be the motion constrained or free, is that the product of the mass by the vector acceleration is equal to the vector force.

Transforming to a set of parallel axes through A, we have:

$$x = x' + x_1, \qquad y = y' + y_1,$$

(4)
$$y' = -\frac{gx'^2}{2v_0^2\cos^2\alpha}.$$

This curve is a parabola with its vertex at A. The height of its directrix above A is $v_0^2\cos^2\alpha/2g$, and hence the height above O of the directrix of the parabola represented by (3) is

$$\frac{v_0^2\sin^2\alpha}{2g} + \frac{v_0^2\cos^2\alpha}{2g} = \frac{v_0^2}{2g}.$$

The result is independent of the angle of elevation α, and so it appears that all the parabolas traced out by projectiles leaving O with the same velocity have their directrices at the same level, the distance of this level above O being the height to which the projectile would rise if shot perpendicularly upward.

EXERCISES

1. Show that the range on the horizontal is

$$R = \frac{v_0^2}{g}\sin 2\alpha,$$

and that the maximum range $\bar{R}$ is attained when $\alpha = 45°$:

$$\bar{R} = \frac{v_0^2}{g}.$$

The height of the directrix above O is half this latter range.

2. A projectile is launched with a velocity of v_0 ft. a sec. and is to hit a mark at the same level and within range. Show that there are two possible angles of elevation and that one is as much greater than 45° as the other is less.

3. Find the range on a plane inclined at an angle β to the horizon and show that the maximum range is

$$R_\beta = \frac{v_0^2}{g}\frac{1}{1+\sin\beta}.$$

4. A small boy can throw a stone 100 ft. on the level. He is on top of a house 40 ft. high. Show that he can throw the stone 134 ft. from the house. Neglect the height of his hand above the levels in question.

5. The best collegiate record for putting the shot was, at one time, 46 ft., and the amateur and world's record was 49 ft. 6 in.

If a man puts the shot 46 ft. and the shot leaves his hand at a height of 6 ft. 3 in. above the ground, find the velocity with which he launches it, assuming that the angle of elevation α is the most advantageous one. *Ans.* $v_0 = 35.87$.

6. How much better record can the man of the preceding question make than a shorter man of equal strength and skill, the shot leaving the latter's hand at a height of 5 ft. 3 in.?

7. Show that it is possible to hit a mark $B: (x_b, y_b)$, provided

$$y_b + \sqrt{x_b^2 + y_b^2} \leqq \frac{v_0^2}{g}.$$

8. A revolver can give a bullet a muzzle velocity of 200 ft. a sec. Is it possible to hit the vane on a church spire a quarter of a mile away, the height of the spire being 100 ft.?

CHAPTER XIV

INFINITE SERIES

1. The Geometric Series. We have met in Algebra the Geometric Progression:

$$a + ar + ar^2 + \cdots,$$

the sum of the first n terms of which is given by the formula:

$$s_n = \frac{a - ar^n}{1 - r}.$$

Suppose, for example, that $a = 1$, $r = \frac{1}{2}$. Then

$$\begin{aligned} s_1 &= 1 &&= 1 \\ s_2 &= 1 + \tfrac{1}{2} &&= 1\tfrac{1}{2} \\ s_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} &&= 1\tfrac{3}{4} \\ s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &&= 1\tfrac{7}{8} \end{aligned}$$

etc.

If we plot on a line the points which represent $s_1, s_2, s_3, \cdots$, we

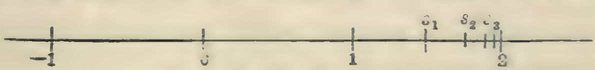


FIG. 127

observe that each new point lies half way between its predecessor and the point 2. Hence it appears that, when n grows larger and larger without limit, s_n approaches 2 as its limit.

Whenever r is numerically less than 1, r^n will approach 0 as its limit when $n = \infty$, and we shall have:

$$\lim_{n=\infty} s_n = \frac{a}{1 - r}.$$

We have here an example of an infinite series, whose value is $a/(1-r)$:

$$\frac{a}{1-r} = a + ar + ar^2 + \cdots, \qquad |r| < 1, \tag{1}$$

and we turn now to the general definition of such series.

2. Definition of an Infinite Series. Let $u_0, u_1, u_2, \cdots$ be any set of values, positive or negative at pleasure. Form the sum:

$$s_n = u_0 + u_1 + \cdot \cdot \cdot + u_{n-1}. \tag{2}$$

When n increases without limit, s_n may approach a limit, U:

$$\lim_{n=\infty} s_n = U.$$

In this case the expression

$$u_0 + u_1 + u_2 + \cdots, \tag{3}$$

called an *infinite series* (or more briefly, a *series*), is said to *converge* and to have the *value* U:

$$U = u_0 + u_1 + u_2 + \cdots.$$

If, on the other hand, s_n approaches no limit, the series (3) is said to *diverge*. For example,

$$1 + 2 + 3 + \cdots,$$
$$1 - 1 + 1 - 1 + \cdots,$$

are illustrations of divergent series. No number is assigned as a value to a divergent series, and such series cannot be used in practice.*

In the early days of the Calculus people spoke of an infinite series as the "sum of an infinite number of terms," or of the value U of a convergent series as "the sum of all its terms." But such words have no meaning. We cannot add up an infinite series, and we do not pretend to do so. We *can* consider

* There are certain classes of divergent series which form an exception to this general statement. Their importance is incomparably less than that of the convergent series here discussed, and their treatment does not belong to the elements of the Calculus.

the *variable* s_n with respect to the question of whether it approaches a limit. Thus the repeating decimal

$$0.333\cdots$$

is to be thought of, not as a sum, but as an *expression*, to which we attach the value

$$\lim_{n=\infty}\left(\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}\cdots+\frac{3}{10^n}\right).$$

The value of the parenthesis is, by § 1,

$$\frac{\frac{3}{10}-\frac{3}{10}\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}}=\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{10^n},$$

and hence the above limit has the value $\frac{1}{3}$. This number, $\frac{1}{3}$, is assigned by definition to the repeating decimal in question as its *value.*

Whenever the series (3) converges, its value can be written in the form

$$U=s_n+r_n,$$

where r_n, called the *remainder*, is a variable which approaches 0 as its limit when n becomes infinite. The value of r_n for any n is the error committed by breaking off the series with its first n terms.

A notation commonly employed for an infinite series, regardless of whether it converges or diverges, is

$$\sum u_n \quad \text{or, more explicitly:} \quad \sum_{n=0}^{\infty} u_n.$$

Thus the infinite geometric series (1) would be written:

$$\sum_{n=0}^{\infty} ar^n.$$

3. Tests for Convergence. Consider the infinite series

$$1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots+\frac{1}{n!}+\cdots, \tag{4}$$

where $n!$ means the product of the first n integers, $1 \cdot 2 \cdot 3 \cdots n$, and is read *factorial n*. Let *

$$s_n = 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdot 3 \cdots n},$$

and form the corresponding sum from the geometric series,

$$S_n = 1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \cdots + \frac{1}{\underbrace{2 \cdot 2 \cdots 2}_{n-1 \text{ factors}}} = 2 - \frac{1}{2^{n-1}} < 2.$$

The first two terms are alike in both series; but from the third term on, those of the first series are less than the corresponding ones in the second. Hence

$$s_n \leqq S_n < 2, \qquad \text{and so} \qquad s_n < 2,$$

no matter how large n be taken. We see, then, that s_n is a variable which always increases as n increases, but which never attains so large a value as 2. To make the situation still more striking, plot the successive values of s_n as points on a line:

$$s_1 = 1, \qquad s_2 = 1.5, \qquad s_3 = 1.667,$$
$$s_4 = 1.708, \qquad s_5 = 1.717, \qquad s_6 = 1.718, \qquad s_7 = 1.718.$$

Thus we see that, when n increases by 1, the point representing s_n always moves to the right, but never advances so far to the right as the point 2. *Hence s_n approaches a limit, which is not greater than* 2, *and the series is convergent.* To judge from the computed values of s_n, the value of the limit to four significant figures is 1.718, — a fact that will be established later.

The reasoning by which we have inferred the existence of a limit in the above example is of prime importance in the theory of infinite series as well as in other branches of analysis. We will formulate it as follows:

* It is convenient to think of the n-th term of this series as $u_n = 1/n!$. Thus the first term will be u_1, not u_0, and the series reads

$$u_1 + u_2 + u_3 + \cdots.$$

An infinite series may begin with a term of any index, as

$$u_8 + u_9 + u_{10} + \cdots.$$

Fundamental Principle. *If s_n is a variable which always increases (or remains unchanged) when n increases:*

(i) $$s_{n'} \geqq s_n, \qquad n' > n;$$

but which never exceeds some definite fixed number, A:

(ii) $$s_n \leqq A,$$

no matter what value n has, then s_n approaches a limit, U:

$$\lim_{n=\infty} s_n = U.$$

The limit U is not greater than A: $\qquad U \leqq A.$

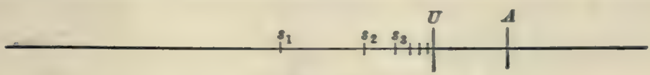


Fig. 128

EXERCISE

State the principle for a variable which is always decreasing, but which remains greater than a certain fixed quantity, and draw the corresponding diagram.

By means of the foregoing principle we can state a simple test for the convergence of an infinite series of positive terms.

Direct Comparison Test for Convergence. *Let*

$$u_0 + u_1 + u_2 + \cdots$$

be a series of positive terms which is to be tested for convergence. If a second series of positive terms already known to be convergent:

$$a_0 + a_1 + a_2 + \cdots,$$

can be found whose terms are greater than or at most equal to the corresponding terms of the series to be tested:

$$u_n \leqq c_n,$$

then the first series converges and its value does not exceed the value of the test-series.

For let
$$s_n = u_0 + u_1 + \cdots + u_{n-1},$$
$$S_n = a_0 + a_1 + \cdots + a_{n-1},$$
$$\lim_{n=\infty} S_n = A.$$

Then since $\qquad S_n < A \qquad$ and $\qquad s_n \leqq S_n,$

it follows that
$$s_n < A.$$

Hence s_n approaches a limit $U \leqq A$, q.e.d.

It is frequently convenient in studying the convergence of a series to discard a few terms at the beginning and to consider the new series thus arising. That the convergence of the latter series is necessary and sufficient for the convergence of the former is evident, since

$$\begin{aligned} s_n &= (u_0 + u_1 + \cdots + u_{m-1}) + (u_m + \cdots + u_{n-1}) \\ &= \qquad\qquad \bar{u} \qquad\qquad + \qquad \bar{s}_{n-m}. \end{aligned}$$

Here m is a fixed number, and so $\bar{u}$ is constant. Hence s_n will converge toward a limit if $\bar{s}_{n-m}$ does, and conversely.

Thus the series

(5) $$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

is seen to converge; for the series obtained by dropping its first term has just been shown to converge. Its value, as will be shown later, is the exponential base,

$$e = 2.71828 \cdots.$$

EXERCISES

Prove the following series to be convergent.

1. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots.$

2. $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots.$

3. $\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots.$

4. $\frac{1}{3!} + \frac{1}{6!} + \frac{1}{9!} + \cdots.$

5. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots.$ Suggestion. Write s_n in the form

$$s_n = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}.$$

6. $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots.$ 7. $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots.$

8. $1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots,\quad p>2.$* 9. $x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots,\ 0\leqq x<1.$

10. $r+r^4+r^9+r^{16}+\cdots,\quad 0\leqq r<1.$

4. Divergent Series. If a series is to converge, then evidently its terms must approach 0 as their limit. For otherwise the points s_n could not cluster about a single point as their limit. Hence we get the following exceedingly simple test for divergence. It holds for series whose terms are positive and negative at pleasure.

If the terms of a series do not approach 0 *as their limit, the series diverges.*

This condition, however, is only sufficient, not necessary, as the following example shows:

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots.$$

If we strike in anywhere in this series and add as many more terms as the number that have preceded:

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n},$$

we get a sum $>\frac{1}{2}$. For, each term just written down is $\geqq 1/2n$, and there are n of them. If, then, we can get a sum greater than $\frac{1}{2}$ out of the series as often as we like, we can get a sum that exceeds a billion, or any other number you choose to name, by adding a sufficient number of terms together. Hence the series diverges in spite of the fact that its terms are growing smaller and smaller and approaching 0 as their limit This series is known as the *harmonic series.*

* It can be shown that this series converges when $p>1$.

A further test for divergence corresponding to the test of § 3 for convergence is as follows:

Direct Comparison Test for Divergence. *Let*

$$u_0 + u_1 + u_3 + \cdots$$

be a series of positive terms which is to be tested for divergence. If a second series of positive terms already known to be divergent:

$$a_0 + a_1 + a_3 + \cdots,$$

can be found whose terms are less than or at most equal to the corresponding terms of the series to be tested:

$$u_n \geqq a_n,$$

then that series diverges.

The proof is similar to that of the test of § 3 for convergence and is left to the student as an exercise.

EXERCISES

Prove the following series to be divergent.

1. $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \cdots.$

2. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots.$

3. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots.$

4. $\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \cdots.$

5. $1 + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \dfrac{1}{4^p} + \cdots, \quad p < 1.$

6. $\dfrac{1}{1 \cdot 5} + \dfrac{1}{2 \cdot 5} + \dfrac{1}{3 \cdot 5} + \cdots.$

5. The Test-Ratio Test. The most useful test for the convergence or divergence of the series we meet most commonly in practice is the following, which holds regardless of whether the terms are positive or negative. It makes use of the ratio of the general term to its predecessor, u_{n+1}/u_n, — the *test-ratio*, as we shall call it.

The Test-Ratio Test. *Let*

$$u_0 + u_1 + u_2 + \cdots$$

be an infinite series and let the limit approached by its test-ratio be denoted by t:

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = t.$$

Then if $|t| < 1$, *the series converges;*

" $|t| > 1$, " " *diverges;*

" $|t| = 1$, *the test fails.*

We shall prove the theorem in this paragraph, so far as it relates to convergence, only for the case that the terms are all positive. Then $t \geqq 0$ and $|t| = t$.

Suppose, then, that $0 < t < 1$. Let γ be chosen between t and 1: $t < \gamma < 1$. Since the variable u_{n+1}/u_n approaches t as its limit, the points representing this variable cluster about the point t and hence ultimately, — *i.e.* from a definite value of n on: $n \geqq m$,—lie to the left of the point γ:

$$\frac{u_{n+1}}{u_n} < \gamma, \qquad n \geqq m.$$

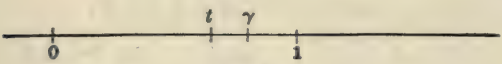


FIG. 129

Now give to n successively the values m, $m+1$, etc.

$$n = m, \qquad \frac{u_{m+1}}{u_m} < \gamma, \qquad u_{m+1} < u_m\gamma;$$

$$n = m+1, \qquad \frac{u_{m+2}}{u_{m+1}} < \gamma, \qquad u_{m+2} < u_{m+1}\gamma < u_m\gamma^2;$$

$$n = m+2, \qquad \frac{u_{m+3}}{u_{m+2}} < \gamma, \qquad u_{m+3} < u_{m+2}\gamma < u_m\gamma^3;$$

. . . .

Hence we see that the terms of the given series, from the term u_m on, do not exceed the terms of the convergent geometric series

$$u_m + u_m\gamma + u_m\gamma^2 + \cdots,$$

and therefore the given series converges.*

* The student should notice that it is not enough, in order to insure convergence, that the test-ratio remain less than unity when $n \geqq m$. Thus for the harmonic series $u_{n+1}/u_n = n/(n+1) < 1$ for all values of n, and yet the series diverges; but here the *limit* of the test-ratio is not less

Secondly let $|t| > 1$, the terms now being either positive or negative. Then, from a definite value of n on,

$$\frac{|u_{n+1}|}{|u_n|} > 1 \quad \text{or} \quad |u_{n+1}| > |u_n|, \qquad n \geqq m,$$

i.e. all later terms are numerically greater than the positive constant $|u_m|$, and so they do not approach 0 as their limit. Hence the series diverges.

Lastly, if $|t| = 1$, we can draw no inference about the convergence of the series, for both convergent and divergent series may have the limit of their test-ratio equal to unity. Thus for the harmonic series, known to be divergent:

$$\frac{u_{n+1}}{u_n} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}, \qquad \lim_{n=\infty} \frac{u_{n+1}}{u_n} = 1;$$

while for the convergent series of § 3, Ex. 6:

$$\frac{u_{n+1}}{u_n} = \left(\frac{n}{n+1}\right)^2, \quad \text{and} \quad \lim_{n=\infty} \frac{u_{n+1}}{u_n} = 1.$$

Example 1. Test for convergence the series

$$\frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots$$

The general term of this series can be written

$$u_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}.$$

Then $$u_{n+1} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot \quad n \quad (n+1)}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)(2n+3)},$$

and hence $$\frac{u_{n+1}}{u_n} = \frac{n+1}{2n+3}.$$

Thus $$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = \lim_{n=\infty} \frac{n+1}{2n+3} = \lim_{n=\infty} \frac{1+\frac{1}{n}}{2+\frac{3}{n}} = \frac{1}{2},$$

or $t = \frac{1}{2}$. Consequently the series converges.

than 1. What is needed for the proof is that the test-ratio should ultimately become and remain less than some *constant* quantity, γ, itself *less than* 1

Example 2. Test the series

$$\frac{x^2}{2^2}+\frac{x^4}{3^2}+\frac{x^6}{4^2}+\cdots.$$

Here, the general term can be written:* $u_n=\dfrac{x^{2n-2}}{n^2}$.

Hence
$$\frac{u_{n+1}}{u_n}=\frac{n^2x^{2n}}{(n+1)^2x^{2n-2}}=\frac{x^2}{\left(1+\frac{1}{n}\right)^2},$$

$$\lim_{n=\infty}\frac{u_{n+1}}{u_n}=x^2,\qquad\text{or}\qquad t=x^2.$$

Thus the series converges when $-1<x<1$, and diverges when $x>1$ or $x<-1$.

If $x=1$ or -1, the test fails. But the series here reduces to

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots,$$

and this series is at once seen to converge; cf. Ex. 7, § 3.

EXERCISES

Test the following series for convergence or divergence.

1. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots.$ *Ans.* Convergent

2. $\frac{1\cdot2}{100^2}+\frac{1\cdot2\cdot3}{100^3}+\frac{1\cdot2\cdot3\cdot4}{100^4}+\cdots.$ *Ans.* Divergent

3. $\frac{1}{2^5}+\frac{3}{2^{10}}+\frac{5}{2^{15}}+\cdots.$

4. $\frac{3}{7}+\frac{5}{7^2}+\frac{7}{7^3}+\cdots.$

5. $\frac{2^{100}}{2}+\frac{3^{100}}{2^2}+\frac{4^{100}}{2^3}+\cdots.$

6. $\frac{3}{5^3}+\frac{3^2}{10^3}+\frac{3^3}{15^3}+\cdots.$

* This will not be the n-th term. It will be the term u_n in the series

$$u_2+u_3+u_4+\cdots.$$

Evidently the test-ratio test applies equally well to a series of the form

$$u_k+u_{k+1}+u_{k+2}+\cdots,$$

where k is any fixed integer.

For what values of x are the following series convergent, and for what values divergent? Draw a figure in each case showing the interval of convergence.

7. $1 + x^2 + x^4 + \cdots.$

8. $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots.$

9. $\frac{x^3}{2} + \frac{x^5}{3} + \frac{x^7}{4} + \cdots.$

10. $1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots.$

11. $\frac{x^2}{3^2} + \frac{x^4}{5^2} + \frac{x^6}{7^2} + \cdots.$

12. $x^2 + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots.$

6. Alternating Series. THEOREM. *Let the terms of an infinite series be alternately positive and negative:*

$$u_0 - u_1 + u_2 - \cdots.$$

If each u is less than or equal to its predecessor:

(i) $$u_{n+1} \leqq u_n,$$

and if

(ii) $$\lim_{n=\infty} u_n = 0,$$

the series converges.

For example

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots.$$

To prove the theorem, denote as usual the sum of the first n terms by s_n. Then, when n is even, $n = 2m$, we have:

$$s_{2m} = (u_0 - u_1) + (u_2 - u_3) + \cdots + (u_{2m-2} - u_{2m-1}).$$

Since each parenthesis is positive, or at worst 0, s_{2m} always increases or remains unchanged when m increases.

If n is odd, $n = 2m+1$,

$$s_{2m+1} = u_0 - (u_1 - u_2) - \cdots - (u_{2m-1} - u_{2m}),$$

and we see that s_{2m+1} steadily decreases or remains unchanged when m increases.

Furthermore, s_{2m} does not exceed the fixed value s_1. For

$$s_{2m} = s_{2m+1} - u_{2m} \leqq s_{2m+1} \leqq s_1.$$

Hence, by the Fundamental Principle of § 3, s_{2m} approaches a limit.

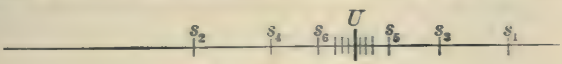


FIG. 130

In like manner it is shown that s_{2m+1} is never less than s_2. For

$$s_{2m+1} = s_{2m} + u_{2m} \geqq s_{2m} \geqq s_2.$$

Hence s_{2m+1} also approaches a limit.

Finally, these limits are equal. For, since

$$s_{2m+1} = s_{2m} + u_{2m}, \qquad \lim_{m=\infty} s_{2m+1} = \lim_{m=\infty} s_{2m} + \lim_{m=\infty} u_{2m},$$

and, by hypothesis, $\lim u_n = 0$. Hence s_n approaches a limit, U, when n becomes infinite passing through both odd and even values, and the series converges, q. e. d.

The Error. It is easily seen that the error made by breaking an alternating series off at any given term does not exceed numerically the value of the last term retained, or the first term dropped. For the distance from s_n to s_{n+1} is u_n, and the distance from s_n to U is not greater.

Further examples of alternating series are the following:

$$(a)\ 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots, \qquad (b)\ -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \cdots,$$

$$(c)\ -\frac{1}{\log 2} + \frac{1}{\log 3} - \frac{1}{\log 4} + \cdots,$$

$$(d)\ x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \leqq x \leqq 1.$$

7. Series of Positive and Negative Terms; General Case. Let

(1) $$u_0 + u_1 + u_2 + \cdots$$

be a series whose terms may be either positive or negative quantities. Form the series of absolute values:

(2) $$|u_0| + |u_1| + |u_2| + \cdots.$$

If this latter series converges, then the original series will converge, too, as we will now prove.

Let the positive terms in (1) be $v_0, v_1, v_2, \cdots$, and form the sum

$$\sigma_m = v_0 + v_1 + \cdots + v_{m-1}.$$

Similarly, let the negative and zero terms in (1) be $-w_0$, $-w_1, -w_2, \cdots$, and form the sum

$$-\tau_p = -w_0 - w_1 - \cdots - w_{p-1}.$$

Let
$$s_n = u_0 + u_1 + \cdots + u_{n-1}.$$

Then, for any given value of n, we can determine m and p so that *

(3)
$$s_n = \sigma_m - \tau_p.$$

Finally, let

$$S_n = |u_0| + |u_1| + \cdots |u_{n-1}|, \qquad \lim_{n=\infty} S_n = A.$$

Evidently,
$$S_n = \sigma_m + \tau_p,$$

and since σ_m and τ_p are positive or zero,

$$\sigma_m \leqq S_n \leqq A, \qquad \tau_p \leqq S_n \leqq A.$$

Hence each of the variables σ_m and τ_p approaches a limit, and thus the series

$$v_0 + v_1 + v_2 + \cdots \qquad \text{and} \qquad w_0 + w_1 + w_2 + \cdots$$

are seen both to converge, if (2) converges.**

From (3) we now infer that s_n also approaches a limit. Hence the series (1) converges, q. e. d.

Example 1. Consider the series

$$\sin x + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \cdots.$$

* If, in particular, for a given n there are no positive terms in s_n, we will understand by σ_0 the number 0. Similarly, if there are no negative or zero terms, we set $\tau_0 = 0$.

** It may happen that the series (1) contains only a finite number of positive terms. The v-series then reduces to an ordinary sum. The reasoning whereby the main theorem is established is not, however, thereby essentially modified. Similarly, if (1) contains only a finite number of negative terms.

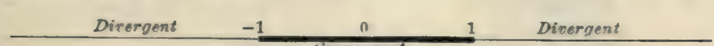
The general term of the series of absolute values is never greater than the corresponding term in the convergent series of Ex. 7, § 3:
$$\frac{|\sin nx|}{n^2} \leq \frac{1}{n^2}.$$
Hence the series of absolute values converges, and so the given series converges for all values of x.

The foregoing condition for the convergence of (1) is sufficient, but not necessary. Thus the series
$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$
has been seen to converge. Here, however, both the v-series and the w-series diverge.

A series (1) whose absolute value series (2) converges is said to be *absolutely* or *unconditionally convergent.* If (1) converges in spite of the fact that (2) diverges, it is said to be *conditionally convergent.*

We can now complete the proof of the theorem of § 5, when $|t| < 1$. Here, the series of absolute values converges; for, its test-ratio has the value
$$\frac{|u_{n+1}|}{|u_n|} = \left|\frac{u_{n+1}}{u_n}\right| \qquad \text{and hence} \qquad \lim_{n=\infty} \frac{|u_{n+1}|}{|u_n|} = |t| < 1.$$
Consequently the u-series converges absolutely.

Example 2. To test the convergence of the series
$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$
Here $$\frac{u_{n+1}}{u_n} = -\frac{n}{n+1}x = -\frac{1}{1+\frac{1}{n}}x, \qquad \lim_{n=\infty}\frac{u_{n+1}}{u_n} = -x,$$
and hence the series converges when $-1 < x < 1$ and diverges outside of this interval.



At the extremities of the interval the test fails. But we see directly that for $x = 1$ the series is a convergent alternating

series; for $x=-1$, the negative of the harmonic series, and hence divergent.

Example 3. The series

$$1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\cdots$$

has for its general term u_n:

$$u_n=\frac{m(m-1)\cdots(m-n+1)}{n!}x^n.$$

If m is a positive integer, the later terms are all 0 and the series reduces to a polynomial, namely, the binomial expansion of $(1+x)^m$. When m is not a positive integer, and $x\neq 0$, the test-ratio is

$$\frac{u_{n+1}}{u_n}=\frac{m-n}{n+1}x, \qquad \text{and} \qquad \lim_{n=\infty}\frac{u_{n+1}}{u_n}=-x.$$

Hence the series converges when $-1<x<1$ and diverges when $|x|>1$. For the determination of whether the series is convergent or divergent at the extremities of the interval of convergence more elaborate tests are necessary. It will be shown later that the value of the series, when $|x|<1$, is always $(1+x)^m$.

EXERCISES

For what values of x are the following series convergent? Indicate the intervals of convergence and divergence each time by a figure.

1. $1+x+2x^2+3x^3+\cdots.$ *Ans.* $-1<x<1.$

2. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots.$

Ans. $-\infty<x<\infty$, *i.e.* for all values of x.

3. $x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\cdots$

4. $x+3x^3+5x^5+\cdots.$

5. $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots.$

6. $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots.$

7. $\frac{x}{2}-\frac{1}{2}\frac{x^2}{2^2}+\frac{1}{3}\frac{x^3}{2^3}-\cdots.$ *Ans.* $-2<x\leqq 2$

8. $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$ 9. $\frac{x}{8} - \frac{1}{3}\frac{x^3}{8^3} + \frac{1}{5}\frac{x^5}{8^5} - \cdots$

10. $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \cdots.$ 11. $x - \frac{x^3}{\sqrt{3}} + \frac{x^5}{\sqrt{5}} - \frac{x^7}{\sqrt{7}} + \cdots$

12. $1 + \frac{1}{\sqrt{2}}\frac{x}{2} + \frac{1}{\sqrt{3}}\frac{x^2}{2^2} + \frac{1}{\sqrt{4}}\frac{x^3}{2^3} + \cdots.$

13. $\frac{1}{\sqrt{3}}\frac{x}{7} - \frac{1}{\sqrt{5}}\frac{x^3}{7^3} + \frac{1}{\sqrt{7}}\frac{x^5}{7^5} - \cdots.$

14. $1 - mx + \frac{m(m-1)}{1\cdot 2}x^2 - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \cdots.$

15. $x + 2!x^2 + 3!x^3 + \cdots.$

Ans. For $x = 0$, and for no other value.

16. $\frac{1}{3}x + \frac{1\cdot 2}{3\cdot 5}x^2 + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}x^3 + \cdots.$ 17. $\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4}\frac{x}{3} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^2}{3^2} + \cdots.$

8. A Series for the Logarithm. Consider the geometric series,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

which converges when $-1 < x < 1$. Integrate each side between the limits 0 and h, where $-1 < h \leqq 1$:

$$\int_0^h \frac{dx}{1+x} = \int_0^h dx - \int_0^h x\,dx + \int_0^h x^2\,dx - \cdots.$$

The value of the left-hand side is

$$\log(1+x)\Big|_0^h = \log(1+h).$$

The integrals on the right are readily computed, and thus we have:

$$(1) \qquad \log(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots.$$

In particular, on setting $h = 1$, we get.

$$(2) \qquad \log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots.$$

There is one flaw in the above reasoning. We have integrated an infinite series as if it were a sum, —*term by term,* as

we say. Now, an infinite series is not a sum, but the *limit* of a sum; and the theorem that the integral of a sum equals the sum of the integrals of the individual terms does not cover the case before us. In fact, not all infinite series can be integrated term by term. Nevertheless, it is easy to show that equation (1) is true. For, we know from algebra that

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}, \quad -1 < r < 1.$$

Hence, setting $r = -x$, we have:

$$1 - x + x^2 - \cdots + (-1)^{n-1}x^{n-1} = 1 - \frac{(-1)^n x^n}{1+x} = \frac{1}{1+x} - \frac{(-1)^n x^n}{1+x}$$

and consequently

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^{n-1}x^{n-1} + \frac{(-1)^n x^n}{1+x}.$$

Thus

$$\int_0^h \frac{dx}{1+x} = \int_0^h dx - \int_0^h x\,dx + \int_0^h x^2\,dx - \cdots$$

$$+ (-1)^{n-1}\int_0^h x^{n-1}dx + (-1)^n \int_0^h \frac{x^n\,dx}{1+x},$$

or

$$\log(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots + (-1)^{n-1}\frac{h^n}{n} + (-1)^n \int_0^h \frac{x^n\,dx}{1+x}.$$

If we can show that the last term approaches 0 as its limit when n becomes infinite, then the sum of the first n terms on the right must approach $\log(1+h)$, and this is precisely the condition that equation (1) be true.

It is easy to give the desired proof. The function $x^n/(1+x)$ is less than x^n when x is positive. Thus the curve

(3) $$y = \frac{x^n}{1+x}$$

lies below the curve

(4) $$y = x^n,$$

and so the area under (3) is less than the area under (4). Hence, if h is positive,

$$\int_0^h \frac{x^n\,dx}{1+x} < \int_0^h x^n\,dx = \frac{h^{n+1}}{n+1},$$

and the value of this last expression does not exceed $1/(n+1)$ when $h \leqq 1$. Thus

$$0 < \int_0^h \frac{x^n\,dx}{1+x} < \frac{1}{n+1}, \qquad 0 < h \leqq 1.$$

But $1/(n+1)$ approaches 0 as n becomes infinite, and this completes the proof when h is positive.

If $h < 0$, let $h' = -h$ and $x' = -x$. Then the numerical value of the integral in question is

$$\int_0^{h'} \frac{x'^n\,dx'}{1-x'}, \qquad 0 < h' < 1.$$

But

$$\frac{x'^n}{1-x'} \leqq \frac{x'^n}{1-h'}, \qquad 0 \leqq x' \leqq h',$$

and hence

$$\int_0^{h'} \frac{x'^n\,dx'}{1-x'} < \frac{1}{1-h'} \int_0^{h'} x'^n\,dx' = \frac{1}{n+1}\,\frac{h'^n}{1-h'} < \frac{1}{(n+1)(1-h')}.$$

This last expression approaches 0 when n becomes infinite, and the proof is now complete.

Incidentally we have obtained the following result, which will be of use to us in computation. *The error committed by breaking the series* (1) *off with n terms is numerically less than the first term neglected, or $h^{n+1}/(n+1)$ when h is positive,* and is numerically less than $h^{n+1}/[(n+1)(1+h)]$ when h is negative:*

* This first result follows also from the general theorem relating to the error in an alternating series.

$$|r_n| < \begin{cases} \dfrac{h^{n+1}}{n+1}, & 0 < h \leqq 1; \\[2ex] \dfrac{|h|^{n+1}}{(n+1)(1-|h|)}, & -1 < h < 0. \end{cases}$$

The process of representing a function by a series, as in formula (1), is known as *developing* or *expanding* the function *into a series.*

EXERCISE

From the equation

$$\int_0^h \frac{dx}{1+x^2} = \tan^{-1} h$$

and the expansion

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots, \qquad -1 < x < 1,$$

obtain the expansion:

$$(5) \qquad \tan^{-1} h = h - \frac{h^3}{3} + \frac{h^5}{5} - \cdots, \qquad -1 \leqq x \leqq 1.$$

9. Computation of Logarithms. At first sight it would seem as if we had gained very little. For if we try, for example, to compute $\log 3$, h must have the value 2, and series (1) diverges. And even when $h = 1$, series (2) converges so slowly that we should have to take thousands of terms to get the value of $\log 2$ even to four places.

But the function $\log x$ has some useful properties, to wit:

$$\log x + \log y = \log xy.$$

Suppose we write 2 in the form:

$$2 = (\tfrac{4}{3})(\tfrac{3}{2}).$$

Then

$$\log 2 = \log \tfrac{4}{3} + \log \tfrac{3}{2}.$$

Setting $1 + h = \frac{4}{3}$, we find that $h = \frac{1}{3}$; thus

$$\log \tfrac{4}{3} = \tfrac{1}{3} - \tfrac{1}{2}(\tfrac{1}{3})^2 + \tfrac{1}{3}(\tfrac{1}{3})^3 - \tfrac{1}{4}(\tfrac{1}{3})^4 + \cdots,$$

and this series converges very well. It is desirable to arrange the computation systematically: *

$\frac{1}{3}$	$= .333\ 33$	$\frac{1}{3}$	$= .333\ 33$
$(\frac{1}{3})^2$	$= .111\ 11$	$\frac{1}{2}(\frac{1}{3})^2$	$= .055\ 56$
$(\frac{1}{3})^3$	$= .037\ 04$	$\frac{1}{3}(\frac{1}{3})^3$	$= .012\ 35$
$(\frac{1}{3})^4$	$= .012\ 35$	$\frac{1}{4}(\frac{1}{3})^4$	$= .003\ 09$
$(\frac{1}{3})^5$	$= .004\ 12$	$\frac{1}{5}(\frac{1}{3})^5$	$= .000\ 82$
$(\frac{1}{3})^6$	$= .001\ 37$	$\frac{1}{6}(\frac{1}{3})^6$	$= .000\ 23$
$(\frac{1}{3})^7$	$= .000\ 46$	$\frac{1}{7}(\frac{1}{3})^7$	$= .000\ 07$
$(\frac{1}{3})^8$	$= .000\ 15$	$\frac{1}{8}(\frac{1}{3})^8$	$= .000\ 02$

From the last column copy off the positive terms by themselves and the negative ones by themselves, and add:

.333 33	.055 56	
.012 35	.003 09	.346 57
.000 82	.000 23	.058 90
.000 07	.000 02	.287 67
.346 57	.058 90	

Thus we have the result:

$$\log \tfrac{4}{3} = .287\ 67$$

The Error. The result is subject to error from two sources: (*a*) the remainder of the series has been dropped; (*b*) the terms retained have been computed only to the accuracy of half a unit in the fifth place.

The error due to (*a*) is less numerically than the first term neglected, or less than 0.000 006; § 8, end.

The error due to (*b*) might conceivably mount up indefinitely near to four units in the fifth place (due to the eight terms added) if the error in each term were large and all the errors worked in the same direction. In practice, such a situation will not occur once in a lifetime, and the result is likely to be

* The student should provide himself with paper ruled in small squares or rectangles, like that used in banks, and write his individual digits in separate compartments.

much more exact than this check on the error would indicate. Still, all that we can *know* in the present case is that, on taking the errors at their worst,

$$.287\ 67 - .000\ 046 < \log\tfrac{4}{3} < .287\ 67 + .000\ 046,$$

or $\log\frac{4}{3} = .287\ 7$, with an error of less than one unit in the fourth place. Cf. Ex. 1 below.

In this paragraph only natural logarithms are considered. From these the denary logarithms are computed; cf. § 11.

EXERCISES

1. Carry the computation of $\log\frac{4}{3}$ two places further out, and determine the certified accuracy of your result.

2. Compute $\log\frac{3}{2}$, carrying the work on each term to five places of decimals.

3. Compute $\log 5$ to four places.

Suggestion: Set $5 = 2^2 \times \frac{5}{4}$.

4. Compute $\log 10$ to four places.

5. Check your result in Question 1 by writing

$$\log\tfrac{4}{3} = -\log\tfrac{3}{4} = -\log(1-\tfrac{1}{4})$$

and setting $h = -\frac{1}{4}$.

6. The corresponding problem with relation to Question 2.

10. On the Computation of Tables. The method of computing any table consists in computing, usually by series, a few relatively widely separated entries to a high degree of accuracy and then obtaining the intermediate entries by interpolation, (second differences, etc.; it is a problem of the Calculus of Finite Differences).

The two demands to be placed on a series which is to be used for computation are: first, that it shall converge rapidly; and secondly, that its terms can be computed with ease. Evidently, these demands are not independent. For, a series may be well adapted for computation, even though a dozen or more terms be needed, if the values of these terms can be written down immediately.

11. More Refined Methods for Computing Logarithms. From the preceding paragraph the student will appreciate the need of computing certain logarithms to a high degree of accuracy. In particular, log 10 must be known to the maximum number of places ever required, for by it we pass from the natural to the denary logarithm, since

$$\log_{10} P = \frac{\log_e P}{\log_e 10}.$$

It is, therefore, of particular interest to see how this logarithm can actually be obtained to a dozen places or more, for the methods of § 9, though suggestive and useful up to four or five places, are not satisfactory much beyond.

The numbers 2 and 5 can be written in the form:*

$$2 = (\tfrac{10}{9})^7 (\tfrac{25}{24})^{-2} (\tfrac{81}{80})^3; \qquad 5 = (\tfrac{10}{9})^{16} (\tfrac{25}{24})^{-4} (\tfrac{81}{80})^7.$$

If, then, we set

$$a = \log \tfrac{10}{9}, \qquad b = \log \tfrac{25}{24}, \qquad c = \log \tfrac{81}{80},$$

we have:

$$\log 2 = 7a - 2b + 3c, \qquad \log 5 = 16a - 4b + 7c,$$

and hence

$$\log 10 = 23a - 6b + 10c.$$

To compute a, write

$$a = -\log \tfrac{9}{10} = -\log(1 - \tfrac{1}{10}),$$

and set $h = -.1$ in the series for $\log(1 + h)$. Similarly

$$b = -\log \tfrac{24}{25} = -\log(1 - \tfrac{1}{25}), \qquad c = \log(1 + \tfrac{1}{80}).$$

* The student should verify these equations by expressing the right-hand sides in terms of prime factors. How these expressions were arrived at, is a question on which we need not enter here; references are given in the articles cited below.

For a lucid account of how to compute a table cf. an article: On the Computation of Logarithms, by Professor James K. Whittemore, *Annals of Mathematics*, 2 ser., vol. 9, 1907-08, p. 1; also the article on Logarithms in the *Encyclopædia Britannica*.

Let it be required to compute log 10 correct to one unit in the eighth place of decimals. Then it is sufficient to compute a, which is multiplied by 23, to ten places, *i.e.* to half a unit in the tenth place. The series,

$$-\log(1-\tfrac{1}{10}) = .1 + \tfrac{1}{2}(.1)^2 + \tfrac{1}{3}(.1)^3 + \cdots,$$

converges relatively slowly; but its terms can be written down with the greatest ease, and so the computation is not arduous. The work should be arranged systematically and blocked out as indicated in § 9. The error due to dropping the remainder of the series can be estimated by the formula of § 8.

It is sufficient to compute b and c each to nine places. The result is:*

$$\log 10 = 2.302\ 585\ 09.$$

The student should now compute $\log_{10} 2$ and $\log_{10} 5$.

A Further Series. In § 20 below the development is obtained:

$$\log \frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right\},$$

convergent when $-1 < x < 1$. If y has any positive value whatever and we set

$$y = \frac{1+x}{1-x}, \qquad \text{then} \qquad x = \frac{y-1}{y+1},$$

and x is numerically less than 1. Hence the logarithm of any number, y, could be computed by means of this series.

12. Computation of π. The computation of π has always been a problem of interest to mathematicians, and even to State Legislatures, one of which came very near enacting a bill whereby its value was settled once for all as $3\frac{1}{7}$!

We learn at school how π can be computed to three figures without great labor by means of inscribed polygons; but the method becomes laborious if pressed far. Our present methods, however, enable us to get ten or a dozen places of decimals with ease.

* The value to fifteen places is (*Tables*, p. 109)

$$\log 10 = 2.30258\ \ 50929\ \ 94046.$$

In § 8, Exercise, the development was obtained:

(1) $$\tan^{-1} h = h - \frac{h^3}{3} + \frac{h^5}{5} - \frac{h^7}{7} + \cdots, \quad -1 \leqq h \leqq 1.$$

Thus, on setting $h = 1$, we have:

$$\pi/4 = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots,$$

— an interesting result when considered as an evaluation of the slowly converging alternating series on the right, but useless as a means of computing π.

It is necessary to make use of properties of the function $\tan^{-1} h$, or its inverse, $\tan \phi$, in order to use the above development with success. One of the methods is as follows. Let*

(2) $$\phi = \tan^{-1} \tfrac{1}{5}, \qquad \text{or} \qquad \tan \phi = \tfrac{1}{5}.$$

Then $$\tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi} = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{10}{24} = \frac{5}{12};$$

(3) $$\tan 4\phi = \frac{\frac{5}{6}}{1 - \frac{25}{144}} = \frac{120}{119},$$

and hence 4ϕ is very nearly equal to $\pi/4$. Now ϕ can readily be computed from (1) to a dozen or more places. So next we set

(4) $$\theta = 4\phi - \tfrac{1}{4}\pi$$

and compute θ as follows:

$$\tan \theta = \frac{\tan 4\phi - 1}{1 + \tan 4\phi} = \frac{\frac{1}{119}}{\frac{239}{119}} = \frac{1}{239};$$

hence $$\theta = \tan^{-1} \tfrac{1}{239}.$$

Thus the series (1), where h is the exceedingly small number $1/239$, will converge rapidly, and we can obtain θ to a dozen places if desired.

* Cf. Chauvenet, *Plane and Spherical Trigonometry*, 3d ed., 1852, p. 125, where the method is still further developed. The analysis of the text is ascribed to Machin.

Knowing ϕ and θ, it remains merely to substitute their values in equation (4) and solve for π:

$$\pi = 16\phi - 4\theta.$$

The value of π to fifteen places is (*Tables*, p. 109)

$$\pi = 3.14159\ 26535\ 89793.$$

EXERCISES

1. Carry through the computation so that the result will be correct to 10 places of decimals.

Note that the first term in the series for θ is most easily determined by long division; but the second term can be determined by means of a four-place table of logarithms.

2. To what degree of precision should you need to know π, in order to compute the length of the equator correct to an eighth of an inch?

13. The Binomial Series. In elementary algebra we meet the Binomial Theorem:

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{1\cdot 2}a^{m-2}b^2 + \cdots \text{ (to } m+1 \text{ terms)},$$

where m is a positive integer. If, in particular, we set $a = 1$, $b = x$, we have:

$$(1)\qquad (1+x)^m = 1 + mx + \frac{m(m-1)}{1\cdot 2}x^2 + \cdots.$$

If m is not a positive integer, we are led to an infinite series, whose convergence has been studied in § 7, Example 3. As stated there, equation (1) still holds when the series converges, *i.e.* when $-1 < x < 1$.

For example, let $m = -1$. The series then reduces to the geometric series

$$1 - x + x^2 - x^3 + \cdots,$$

whose value is $1/(1+x)$, and equation (1) is thus directly verified.

The following cases are particularly important in practice.

$$(2)\qquad \frac{1}{\sqrt{1+x}} = 1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\cdots;$$

$$(3)\qquad \frac{1}{\sqrt{1-x^2}} = 1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\cdots;$$

$$(4)\qquad \sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{2\cdot 4}x^2+\frac{1\cdot 3}{2\cdot 4\cdot 6}x^3-\cdots;$$

$$(5)\qquad \sqrt{1-x^2} = 1-\frac{1}{2}x^2-\frac{1}{2\cdot 4}x^4-\frac{1\cdot 3}{2\cdot 4\cdot 6}x^6-\cdots.$$

Each of these series converges in the interval $-1<x<1$. Moreover, the error made by breaking off with a given term can be estimated, when the series is an alternating series, by the usual rule, § 6. When, however, the terms in the remainder series all have the same sign, notice that each coefficient is numerically less than its predecessor. Hence, on replacing each coefficient by the coefficient of the first term in the remainder series, we have a geometric series whose value is greater numerically than the value of the error.

EXERCISES

1. Show that, when $m=-2$, equation (1) becomes

$$\frac{1}{(1+x)^2}=1-2x+3x^2-4x^3+\cdots.$$

2. Obtain the result of Question 1 by differentiation, starting with the equation

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots,$$

and assuming that a power series may be differentiated like a polynomial, *i e. term by term.*

3. Obtain in two ways a development for $\dfrac{1}{(1+x)^3}$.

4. From formula (3) obtain the development:

$$\sin^{-1}h=h+\frac{1}{2}\frac{h^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{h^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{h^7}{7}+\cdots,$$

5. Show that

$$\frac{\pi}{6}=\frac{1}{2}+\frac{1}{2}\frac{(\frac{1}{2})^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{(\frac{1}{2})^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{(\frac{1}{2})^7}{7}+\cdots.$$

6. Show that, to four places of decimals,

$$\sqrt{0.99}=0.9950, \qquad \text{and} \qquad \sqrt{1.01}=1.0050.$$

7. Compute to four places of decimals

$$(a)\ \frac{1}{.99}; \qquad (b)\ \frac{1}{\sqrt{.99}}; \qquad (c)\ \frac{1}{1.01}; \qquad (d)\ \frac{1}{\sqrt{1.01}}.$$

8. If h is numerically small, show that $\sqrt{1+h}$ and $\sqrt{1-h}$ are represented approximately by

$$1+\tfrac{1}{2}h \qquad \text{and} \qquad 1-\tfrac{1}{2}h.$$

9. Observing that $2=(\frac{7}{5})^2\,\frac{50}{49}$, compute $\sqrt{2}$ to six places of decimals.

10. Observing that $2=(\frac{5}{4})^3\,\frac{128}{125}$, compute $\sqrt[3]{2}$ to six places of decimals.

14. Arc of an Ellipse. Let the ellipse be represented parametrically: *

$$x=a\sin\phi, \qquad y=b\cos\phi.$$

Then $ds^2=[a^2\cos^2\phi+b^2\sin^2\phi]d\phi^2=[a^2-(a^2-b^2)\sin^2\phi]\,d\phi^2$ and

$$(1) \qquad s=a\int_0^\phi\sqrt{1-e^2\sin^2\phi}\,d\phi,$$

where $e=\sqrt{a^2-b^2}/a$ denotes the eccentricity. This integral is known as an *Elliptic Integral.* Its value cannot be found in the usual way, since the indefinite integral cannot be expressed in terms of the elementary functions. Its value can, however, be found by the aid of infinite series. Since $0<e<1$, equation (5), § 13, will hold if we set $x=e\sin\phi$:

$$\sqrt{1-e^2\sin^2\phi}=$$

$$1-\frac{1}{2}e^2\sin^2\phi-\frac{1}{2\cdot 4}e^4\sin^4\phi-\frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\sin^6\phi+\cdots.$$

* Cf. *Analytic Geometry*, pp. 119 and 306.

On integrating each side of this equation, we have:*

$$(2)\qquad s = a\left[\phi - \frac{1}{2}e^2\int_0^\phi \sin^2\phi\, d\phi - \frac{1}{2\cdot 4}e^4\int_0^\phi \sin^4\phi\, d\phi - \cdots\right].$$

These integrals are given by formula 263 in Peirce's *Tables* In particular, the length of a quadrant corresponds to the value $\phi = \pi/2$, and by formula 483,

$$\int_0^{\frac{\pi}{2}} \sin^n\phi\, d\phi = \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots\quad n}\,\frac{\pi}{2},\quad n\text{, an even integer.}$$

The elliptic integral then becomes the integral known as the Complete Elliptic Integral of the Second Kind; it is denoted by E:

$$(3)\qquad E = \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2\phi} =$$

$$\frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{e^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{e^6}{5} - \cdots\right].$$

Hence the total perimeter, P, of the ellipse is

$$(4)\qquad P = 4aE.$$

If $e = 0$, the ellipse becomes a circle, and $E = \pi/2$, $P = 2\pi a$.

Both E and the integral of (1) are tabulated; cf. Peirce, *Tables*, pp. 121, 123.

EXERCISES

1. Compute the perimeter of an ellipse whose major axis is 10 cm. and whose minor axis is $5\sqrt{2}$ cm., correct to four places of decimals.

2. A tomato can, 4 inches in diameter, from which the top and bottom have been removed, is bent into the form of an

* The justification for this step, *i.e.* the proof that the series before us can be integrated term by term, cannot be given here. It belongs to a later stage in analysis.

elliptic cylinder, whose minor axis is 10 per cent. shorter than the major axis. How large should the new top and bottom be made? Compute the answer correct to one sixty-fourth of an inch.

15. Simple Pendulum. It is shown in Mechanics that the period of a complete oscillation of a simple pendulum of length l is given by the formula:

$$(1)\qquad T=4K\sqrt{\frac{l}{g}},\qquad K=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},\qquad k=\sin\frac{\alpha}{2},$$

where α denotes the maximum inclination of the pendulum to the vertical. K is known as the Complete Elliptic Integral of the First Kind. It can be evaluated as follows. In formula (3) of § 13, set $x=k\sin\phi$. Then

$$\frac{1}{\sqrt{1-k^2\sin^2\phi}}=1+\frac{1}{2}k^2\sin^2\phi+\frac{1\cdot 3}{2\cdot 4}k^4\sin^4\phi+\cdots.$$

Integrating and reducing as in § 14, we find:

$$(2)\qquad K=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2k^6+\cdots\right].$$

The function is tabulated in Peirce's *Tables*, p. 121.

When α is small, an approximation to K is obtained by setting $k=0$. Then $K=\pi/2$, and

$$(3)\qquad T=2\pi\sqrt{\frac{l}{g}},$$

the usual pendulum formula. In the course of a day, however, the second term becomes appreciable, even though α be only 4°. For then

$$k=\sin 2°=.0349,\qquad k^2=.001\,218$$

and if T denotes the period as given by (3), T', the period as given by (1), we have, to six places of decimals,

$$T'=1.000\,305\ T.$$

Let N and N' be the corresponding number of oscillations in a day. Then

$$NT = N'T', \qquad \frac{N}{N'} = \frac{T'}{T} = 1.000\,305,$$

$$\frac{N}{N'} - 1 = .000\,305, \qquad N - N' = .000\,305\,N',$$

and the discrepancy is given by this last expression. Thus for a seconds pendulum, $T' = 2$, $N' = \frac{24 \times 60 \times 60}{2} = 43{,}200$, and $N - N' = 13$ seconds in a day.

16. Approximate Formulas in Applied Mathematics. It is often possible to replace a complicated formula in applied mathematics by a simpler one which is still correct within the limits of error of the observations.*

The Coefficient of Expansion. By the coefficient of linear expansion of a solid is meant the ratio

$$\alpha = \frac{l' - l}{l} \bigg/ (t' - t),$$

where l is the length of a piece of the substance at temperature $t°$, l' the length at temperature $t'°$. The coefficient of cubical expansion is defined similarly as

$$\beta = \frac{V' - V}{V} \bigg/ (t' - t),$$

where V, V' stand for the volumes at $t°$, $t'°$ respectively. Then

$$\frac{V' - V}{V} = \frac{l'^3 - l^3}{l^3},$$

as is at once clear if we consider a cube of the substance, the length of an edge being l at $t°$, l' at $t'°$; in fact, the equation holds for any two similar solids. The accurate expression for α in terms of β is as follows:

$$\frac{l'}{l} = 1 + \alpha(t' - t) = \sqrt[3]{1 + \beta(t' - t)},$$

$$\alpha(t' - t) = \sqrt[3]{1 + \beta(t' - t)} - 1 = \tfrac{1}{3}\beta(t' - t) - \tfrac{1}{9}\beta^2(t' - t)^2 + \cdots,$$

* See Kohlrausch, *Physical Measurements*, §§ 1–6.

Since β is small,—usually less than .0001,—the error made by neglecting the terms of the series subsequent to the first is less than the errors of observation and hence we may assume without any loss of accuracy that

$$\alpha = \tfrac{1}{3}\beta, \qquad \beta = 3\alpha.$$

Double Weighing. Show that, if the apparent weight of a body when placed in one scale pan is p_1, when placed in the other scale pan, p_2 (the difference being due to a slight inequality in the lengths of the arms of the balance), the true weight p is given with sufficient accuracy by the formula:

$$p = \tfrac{1}{2}(p_1 + p_2).$$

Errors of Observation. In an experimental determination of a physical magnitude it is important to know what effect an error in an observed value will have on the final result. For example, let it be required to determine the radius of a capillary tube by measuring the length of a column of mercury contained in the tube, and weighing the mercury. From the formula

$$w = \pi r^2 l\rho,$$

where w denotes the weight of the mercury in grammes, l the length of the column in centimeters, ρ the density of the mercury ($= 13.6$), and r the radius of the tube, we get

$$r = \sqrt{\frac{w}{\pi\rho l}} = .153\sqrt{\frac{w}{l}}.$$

Now the principal error in determining r arises from the error in observing l. Let l be the true value, $l' = l + e$ the observed value, of the length of the column; r the true value, $r' = r + E$ the computed value of the radius. Then E is the error in the result arising from the error of observation e, the error in observing w being assumed negligible. Hence

$$E = .153\sqrt{\frac{w}{l'}} - .153\sqrt{\frac{w}{l}} = .153\sqrt{\frac{w}{l}}\left[\left(1 + \frac{e}{l}\right)^{-\frac{1}{2}} - 1\right]$$

$$= r\left(-\frac{1}{2}\frac{e}{l} + \frac{3}{8}\frac{e^2}{l^2}\cdots\right).$$

Since e is small compared with l, we get a result sufficiently accurate by taking only the first term; and hence, approximately,

$$E = -\tfrac{1}{2} r \cdot \frac{1}{l} \cdot e.$$

Thus for a given error in observing l, the error in the computed value of r is inversely proportional to the length of the column of mercury used,—a result not *a priori* obvious, for r itself is inversely proportioned only to $\sqrt{l}$.

In the foregoing, e and E are the *absolute errors*. The *relative errors*, e/l and E/r, are connected by the relation:

$$\frac{E}{r} = -\tfrac{1}{2}\frac{e}{l}.$$

Thus the relative error in the result is half as great as the relative error in the observed value, and of opposite sign.*

EXERCISES

The student should observe each time whether the absolute error or the relative error, or both, are of physical interest.

1. If, in the example last discussed, there is an error of h in measuring w, show that the error in the result will be

$$E = \tfrac{1}{2} r \cdot \frac{1}{w} \cdot h.$$

What will the relative error be?

2. If, in the above example, the errors in l and w are both considered, show that

$$E = -\tfrac{1}{2} r \frac{1}{l} e + \tfrac{1}{2} r \frac{1}{w} h.$$

What will be the relative error?

* The results obtained here can also be obtained by means of differentials. But the differential formulas yield no indication as to the size of the quantity which is thrown away, whereas the method of series gives an upper limit for this quantity in explicit form.

3. The value of g is determined by a *seconds pendulum,*— *i.e.* by a simple pendulum which beats approximately 1 sec. in each half excursion,—with the aid of formula (3), § 15. Show that an error e in observing T will give rise to an error E in the computed value of g, which is about $10l$ times as great as e, and has the opposite sign.

4. Obtain a formula for the error in the foregoing problem, when the errors in observing T and l are both considered.

5. An engineer surveys a field, using a chain that is incorrect by one tenth of one per cent. of its length. Show that the error thus arising in the determination of the area of the field will be two tenths of one per cent. of the area.

Suggestion: Consider first a triangle, and assume that the error in measuring the angles is negligible.

6. Show that, in using the binomial expansion for $\sqrt[3]{1+x}$, the error made by breaking off with the term in x^n is less numerically than the coefficient of that term (or indeed of the next term) multiplied by $x^{n+1}/(1-x)$.

7. Generalize for $(1+x)^m$, when $m \geqq -1$.

17. Continuation: Pendulum Problems. A clock regulated by a pendulum is located at a point (A) on the earth's surface. If it is carried to a neighboring point (B), h feet above the level of (A), show that it will lose $\frac{1}{244}h$ seconds a day, *i.e.* one second for every 244 feet of elevation.

The number of seconds N that the clock registers in 24 hours is directly proportional to the number of beats of its pendulum, and hence inversely proportional to the period T of the oscillation of the pendulum. Hence by formula (3) of § 15

$$\frac{N'}{N}=\frac{T}{T'}=\sqrt{\frac{g'}{g}},$$

where the unprimed letters refer to the location (A), the primed letters to (B). If the clock was keeping true time at (A), then $N=86{,}400$.

Furthermore, the attraction of the earth or a particle situated at an appreciable distance beyond its surface is inversely proportional to the distance of the particle from its centre; cf. Chap. XIII, § 7. Hence

$$\frac{g'}{g}=\frac{R^2}{(R+h)^2},$$

where R denotes the length of the radius of the earth. It follows, then, that

$$1-\frac{N'}{N}=1-\sqrt{\frac{g'}{g}}=\frac{h}{R+h},$$

$$N-N'=N\frac{h}{R+h}=N\frac{h}{R}-N\frac{h^2}{R^2}+N\frac{h^3}{R^3}-\cdots.$$

If h does not exceed 4 miles, $h/R<.001$, $h^2/R^2<.000\,001$, and the first term of the series gives $N-N'$ correct to seconds:

$$N-N'=\tfrac{1}{244}h.$$

EXERCISES

1. The summit of Mt. Washington is 6226 feet above sea level. How many seconds a day will a clock lose that keeps accurate time in Boston Harbor, if carried to the summit of the mountain?

2. Show that the number of seconds lost in a day by a clock regulated by a pendulum is given by the formula:

$$n=43{,}200\,\alpha t,$$

where α denotes the coefficient of linear expansion and t, the rise in temperature.

For brass, $\alpha=.000\,019$, t being measured in degrees centigrade. Thus for a brass pendulum $n=.82t$, and a rise in temperature of 5° causes the clock to lose a little over 4 seconds a day.

3. It is shown in Mechanics that the force of attraction at any point within the earth is directly proportional to the distance from the centre. A pendulum which beats seconds on

the surface of the earth is observed to lose one second a day when carried to the bottom of a mine. How deep is the mine?

Ans. 489 ft.

4. The weights of an astronomical clock exert, through faulty construction of the clock, a greater driving force when the clock has just been wound up than when it has nearly run down, and thus increase the amplitude of the pendulum from 2° to 2° 4′ on each side of the vertical. Show that, if the clock keeps correct time when it has nearly run down, it will lose at the rate of about .4 of a second a day when it has just been wound up.

5. Show that an arc of a great circle of the earth, $2\frac{1}{2}$ miles long, recedes 1 foot from its chord.

6. Show that, in levelling, the correction for the curvature of the earth is 8 in. for one mile. How much is it for two miles?

7. "A ranchman 6 feet, 7 inches tall, standing on a level plain, agrees to buy at $7 an acre all the land in sight. How much must he pay? Given 640 acres make a square mile" Admission Examination in Solid Geometry, June, 1895.

Show that, if the candidate had assumed the altitude of the zone in sight to be equal to the height of the ranchman's eyes above the ground, — 6 feet, 4 inches, let us say, — and had made no other error in his solution, his answer would have been 4 cents too large.

8. A man is standing on the deck of a ship and his eyes are h feet above the sea level. If D denotes the shortest distance, measured in miles, of a ship away whose masts and rigging he can see, but whose hull is invisible to him, and if h_1 denotes the height, measured in feet, to which the hull rises out of the water, show that, if refraction can be neglected,

$$D = 1.23(\sqrt{h} + \sqrt{h_1}).$$

If $h = h_1 = 16$ ft., $D = 10$ miles (nearly).

9. The focal length f of a lens is given by the formula

$$\frac{1}{f}=\frac{1}{p_1}+\frac{1}{p_2},$$

where p_1 and p_2 denote two conjugate focal distances. Obtain a simpler approximate formula for f that will answer when p_1 and p_2 are nearly equal.

10. Two nearly equal but unknown resistances, A and B, form two arms of a Wheatstone's Bridge. A standard box of coils and a resistance x to be measured form the other two arms. A balance is obtained when the standard rheostat has a resistance of r ohms. When, however, A and B are interchanged, a balance is obtained when the resistance of the rheostat is r' ohms. Show that, approximately,

$$x=\tfrac{1}{2}(r+r').$$

18. Taylor's Theorem. There is a theorem of a very general nature and of wide reaching importance which says that such functions as are studied in the Calculus, — and many others, — can, under suitable restrictions, be represented by power series. The formula is as follows:

$$(1)\quad f(x_0+h)=f(x_0)+f'(x_0)h+f''(x_0)\frac{h^2}{2!}+f'''(x_0)\frac{h^3}{3!}+\cdots$$

or, since $x=x_0+h$ and $h=x-x_0$,

$$(2)\quad f(x)=$$

$$f(x_0)+f'(x_0)(x-x_0)+f''(x_0)\frac{(x-x_0)^2}{2!}+f'''(x_0)\frac{(x-x_0)^3}{3!}+\cdots,$$

where $f'(x)$, $f''(x)$, etc. denote the successive derivatives; cf. Chap. X, § 1.

Here, x_0 is an arbitrarily chosen value of x. Obviously, it must be a value for which all the derivatives have definite values. Thus, if $f(x)=x^{\frac{3}{2}}$, x_0 cannot be 0. Although this condition is not sufficient to insure the validity of the formula, still in the cases which arise in practice, exceptions are rare, — if indeed, they ever occur.

Formulas (1) and (2) are known as *Taylor's Theorem.* The particular case that $x_0 = 0$ gives

$$(3) \qquad f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots,$$

and this formula is known as *Maclaurin's Theorem.* Brook Taylor and Colin Maclaurin were mathematicians of the early part of the eighteenth century. The proof of the theorem dates, however, from the nineteenth century, and will be given in § 22.

Example. Let $f(x) = x^m$ and let $x_0 = 1$. Then

$$f'(x) = mx^{m-1}, \qquad f''(x) = m(m-1)x^{m-2},$$

$$f'''(x) = m(m-1)(m-2)x^{m-3}, \qquad \text{etc.};$$

$$f(1) = 1, \qquad f'(1) = m, \qquad f''(1) = m(m-1),$$

$$f'''(1) = m(m-1)(m-2), \qquad \text{etc.}$$

Thus formula (1) gives:

$$(1+h)^m = 1 + mh + \frac{m(m-1)}{1\cdot 2}h^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}h^3 + \cdots,$$

the Binomial Series for an arbitrary m, commensurable or incommensurable, positive or negative.

EXERCISES

1. If $f(x) = x^m$, and if $x_0 > 0$ and $x_0 + h > 0$, show that (1) becomes:

$$(x_0+h)^m =$$

$$x_0^m + mx_0^{m-1}h + \frac{m(m-1)}{1\cdot 2}x_0^{m-2}h^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x_0^{m-3}h^3 + \cdots$$

Prove that the series converges if $-x_0 < h < x_0$.

2. Deduce from (1) the series for $\log(1+h)$, Formula (1), § 8.

3. Apply Maclaurin's Theorem to the function e^x; cf. Formula (1), § 19.

4. The same for $\sin x$.

5. The same for $\cos x$.

19. Series for e^x, $\sin x$, $\cos x$. If we apply Maclaurin's Theorem, considered merely as Formula (3) § 18, to these functions, we obtain the following formulas. The proof that these series converge to the value of the function will be given in § 23.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots; \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots; \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots. \tag{3}$$

In equations (2) and (3), x is measured in radians. For angles less than half a right angle, $x < \pi/4 < .8$, and the series converge rapidly. We note the value:

$$\frac{\pi}{180} = .01745\ 32925\ 19943.$$

EXERCISES

1. Compute the value of $e^{.06}$ correct to six places of decimals.

2. Compute $\sin 6°$ correct to six places of decimals.

3. Show that the error made by replacing $\sin x$ by x is less than one unit in the third place of decimals, provided x is numerically less than .18.

4. How great may x be numerically, if the error made by replacing $\cos x$ by $1 - \frac{1}{2}x^2$ is not to exceed one unit in the fifth place of decimals? *Ans.* $|x| < .1245$.

5. What is the largest number of terms that ever would have to be used in order to compute $\sin x$ correct to six places of decimals from series (2) if x corresponds to a positive angle less than 45°? *Ans.* Four.

6. The same for $\cos x$ and series (3).

20. Algebraic Operations with Series. Out of the developments above considered, — namely, those for

$$\log(1+x), \qquad \tan^{-1}x, \qquad \sin^{-1}x, \qquad (1+x)^m,$$
$$e^x, \qquad \sin x, \qquad \cos x,$$

other series can be obtained by operating with power series as if they were polynomials.

Addition. If

$$(1) \qquad \begin{cases} U = u_0 + u_1 + u_2 + \cdots \\ V = v_0 + v_1 + v_2 + \cdots \end{cases}$$

are any two convergent series, it is readily shown that their sum, $U+V$, can be expressed by the series

$$(2) \qquad U + V = u_0 + v_0 + u_1 + v_1 + u_2 + \cdots,$$

and a similar theorem holds for the difference, $U - V$.

Thus, for example, from the development

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,$$

we infer, on replacing x by $-x$, that

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

Hence, on subtracting the second series from the first, we have:

$$\log\frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right\},$$

and this series converges when $-1 < x < 1$.

Multiplication. If both the series (1) converge absolutely, their product is given by the absolutely convergent series

$$(3) \qquad UV = u_0v_0 + u_0v_1 + u_1v_0 + u_0v_2 + u_1v_1 + u_2v_0 + \cdots.$$

The proof of this and the subsequent theorems about series must be postponed until the whole theory of infinite series can be treated systematically.

In particular, let $f(x)$ and $\phi(x)$ be two functions represented by power series:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots, \qquad -r < x < r;$$
$$\phi(x) = b_0 + b_1x + b_2x^2 + \cdots, \qquad -s < x < s.$$

Then their product is given by the power series

(4) $f(x)\phi(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots.$

This latter series converges at least throughout the smaller of the two intervals of convergence of the given series.

If $\phi(x) = f(x)$, we have

$$(5) \qquad \{f(x)\}^2 = a_0^2 + 2a_0 a_1 x + (a_0 a_2 + 2a_1^2)x^2 + \cdots.$$

Division. It is even true that one power series can be divided by another, just as if they were both polynomials. For example,

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots};$$

$$\begin{array}{r|l} & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\ \hline 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \cdots & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\ & x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\ \hline & \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\ & \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\ \hline & \frac{2}{15}x^5 + \cdots \end{array}$$

We can obtain in this way as many terms in the development of $\tan x$ as we wish, although the law of the series does not become obvious.

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots.$$

The resulting power series will converge throughout a certain interval, whose extent cannot, however, be determined by any simple rule.

In particular, one or both of the given power series may be a polynomial.

EXERCISES

Deduce the following developments, obtaining the law of the series when convenient; otherwise computing a few terms.

1. $\dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots.$

2. $\dfrac{e^x + e^{-x}}{2} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots.$

3. $\dfrac{\sin x}{x} = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \cdots.$

4. $\dfrac{\log(1+x)}{x} = 1 - \dfrac{x}{2} + \dfrac{x^2}{3} - \dfrac{x^3}{4} + \cdots.$

5. $e^x \sin x = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots.$

6. $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots.$

7. $\cot x = \dfrac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \cdots.$

8. $\csc x = \dfrac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \cdots.$

9. $\dfrac{\log(1+x)}{1+x} = x + \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{25}{12}x^4 + \cdots.$

10. $\dfrac{1+x}{1+x+x^2} = 1 - x^2 + x^3 - x^5 + x^6 - x^8 + x^9 - \cdots.$

21. Integration by Means of Series. We have already seen in §§ 14, 15 that, by means of series, some integrals can be evaluated which cannot be computed by the earlier methods. It is not true that any convergent series of continuous functions can be integrated term by term, or even that such a series necessarily represents a continuous function. But it can be shown that *a power series always represents a continuous function throughout its interval of convergence, and that it can be integrated term by term, and also differentiated term by term:*

(1) $$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots, \qquad -r < x < r;$$

(2) $$\int_0^h f(x)\,dx = a_0 h + a_1 \frac{h^2}{2} + a_2 \frac{h^3}{3} + \cdots;$$

(3) $$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots,$$

where h has any value in the interval of convergence, $(-r, r)$. Equation (2) holds even when $h = r$ (or $h = -r$) and the series (1) diverges for this value, provided that (2) converges; cf. for example equation (2), § 8.

If, in equation (1), x is replaced by a continuous function of t,

$$x = \phi(t), \qquad a \leqq t \leqq b,$$

the new series can also be integrated term by term:

$$(4) \quad \int_a^b f[\phi(t)]\,dt = a_0(b-a) + a_1\int_a^b \phi(t)\,dt + a_2\int_a^b \{\phi(t)\}^2 dt + \cdots,$$

provided that* $-r < \phi(t) < r$.

EXERCISES

Prove the following equations to be correct.

1. $\displaystyle\int_0^\pi \frac{\sin x}{x}\,dx = 1.852.$

2. $\displaystyle\int_0^1 \frac{e^x - e^{-x}}{x}\,dx = 2.114\,502$

3. $\displaystyle\int_0^{1.1} e^{-x^2}dx = 0.7801.$

4. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1-\cos x}{x^2}\,dx = 0.7343.$

5. $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^4}} = 0.503.$

6. $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{dx}{\sqrt{1+x^4}} = 0.247.$

7. $\displaystyle\int_2^{10} \frac{dx}{\sqrt{1+x^4}} = 0.486.$ Suggestion. Set $x = 1/t$.

8. $\displaystyle\int_0^1 \frac{\log(1+x)}{x}\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$

9. $\displaystyle\int_0^x e^{-x^2}dx = x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots.$

10. $\displaystyle\int_0^1 \frac{x^{a-1}dx}{1+x^b} = \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \cdots, \quad (0 < a, \ 0 < b).$

* Equation (4) holds even when (1) diverges for $x = r$, provided that the series (4) converges and $-r < \phi(t) < r$ for values of $a \leqq t < b$, if $\phi(t)$ approaches the limit r monotonically as t approaches b.

22. Proof of Taylor's Theorem with the Remainder. The following formula is known as Taylor's Theorem with the Remainder:

$$(1)\quad f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\cdots$$
$$+\frac{h^n}{n\,!}f^{(n)}(a)+\frac{h^{n+1}}{(n+1)\,!}f^{(n+1)}(a+\theta h),$$

where θ is an unknown number lying between 0 and 1;

$$0<\theta<1.$$

The theorem is true whenever $f(x)$ and its first $n+1$ derivatives are continuous in the interval $a \leqq x \leqq a+h$. Its proof is based on

ROLLE'S THEOREM. *Let $\phi(x)$ be continuous in the interval $a \leqq x \leqq b$, and let $\phi(x)$ vanish at the extremities of the interval:*

$$\phi(a)=0, \qquad \phi(b)=0.$$

Furthermore, let $\phi(x)$ have a derivative, $\phi'(x)$, at every interior point, x, of the interval; $a<x<b$. Then $\phi'(x)$ vanishes for at least one of these points, X:

$$\phi'(X) = 0, \qquad a<X<b.$$

Geometrically, this theorem seems well-nigh self-evident, for the graph of the function,

$$y=\phi(x),$$

meets the axis of x at the points $x=a$ and $x=b$, and is a continuous curve. It must, therefore, have a highest point or a lowest point (or both) in between, and so the graph must be as indicated in Fig. 131. Since $\phi(x)$ has a derivative, the slope of the curve at such a point will be 0:

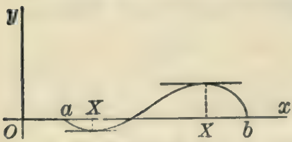


FIG. 131

$$\phi'(x)=0, \qquad x=X.$$

Such a graph as is shown in Fig. 132 is impossible, since the corresponding function has no derivative in the points in which it has a maximum or a minimum.

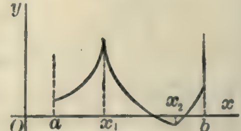


FIG. 132

The importance of Rolle's Theorem in analysis lies in the fact that it can be proven *arithmetically*, without any use of geometry.

Proof of (1). Let R be determined by the equation

$$(2)\ f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^{(n)}(a)+R.$$

We proceed to obtain an expression for R which will be useful in practice. This can be done in a variety of ways, the most important being the following. Let P be determined by the equation:

$$R=\frac{h^{n+1}}{(n+1)!}P.$$

Then (2) can be written in the form:

$$(3)\ f(a+h)-f(a)-hf'(a)-\cdots-\frac{h^n}{n!}f^{(n)}(a)-\frac{h^{n+1}}{(n+1)!}P=0.$$

We now proceed to form the following function of x:

$$(4)\ \phi(x)=f(A)-f(x)-(A-x)f'(x)-\frac{(A-x)^2}{2!}f''(x)-\cdots$$
$$-\frac{(A-x)^n}{n!}f^{(n)}(x)-\frac{(A-x)^{n+1}}{(n+1)!}P,$$

where $A=a+h$.* This function is continuous in the interval $a \leqq x \leqq A$, and it has a derivative. Moreover, it obviously vanishes when $x=A$; but it vanishes also when $x=a$. For, $A-a=h$, and the right hand side of (4) now reduces to the left hand side of (3).

Thus this function $\phi(x)$ is seen to satisfy all the conditions of Rolle's Theorem, and hence, for a suitable value, $x=X$,

$$\phi'(X)=0, \qquad a<X<A.$$

* Since it took the race two centuries to develop this formula after the Calculus was invented, the student will not be surprised that the reasons which underlie it cannot be given him in a few words. Let him accept it as a *deus ex machina*. The formula once written down, the reasoning which follows is simple.

The derivative of $\phi(x)$ is easily computed:

$$\phi'(x) = -f'(x) + f'(x) - (A-x)f''(x) + (A-x)f''(x) - \cdots$$
$$-\frac{(A-x)^n}{n!}f^{(n+1)}(x) + \frac{(A-x)^n}{n!}P,$$

or
$$\phi'(x) = \frac{(A-x)^n}{n!}[-f^{(n+1)}(x) + P].$$

Since this function is to vanish for X, the bracket must vanish for this value, and so

$$P = f^{(n+1)}(X).$$

On writing X in the form $X = a + \theta h$, where $0 < \theta < 1$, and substituting in (2) for R the value thus obtained, we see that

(5)
$$R = \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta h),$$

and thus we have the proof of the truth of (1), when $h > 0$. If $h < 0$, the only change in the proof consists in reversing the signs of inequality: $A \leqq x \leqq a, \quad A < X < a.$

EXERCISE

Set $R = hP$ and thus obtain the following form for the remainder:

(6)
$$R = \frac{(1-\theta)^n h^{n+1}}{n!}f^{(n+1)}(a+\theta h), \qquad 0<\theta<1.$$

23. Proof of the Development of e^x, $\sin x$, $\cos x$. The function e^x can be developed by Taylor's Theorem about the point $a = 0$. Here

$$f(x) = e^x, \qquad f'(x) = e^x, \quad \cdots \quad f^{(n)}(x) = e^x,$$
$$f(0) = 1, \qquad f'(0) = 1, \quad \cdots \quad f^{(n)}(0) = 1,$$

and the remainder R as given by (5) has the form:

$$R = \frac{h^{n+1}}{(n+1)!}e^{\theta h}.$$

If $\quad h < 0$, then $e^{\theta h} < 1$, $\quad$ and $\quad |R| < \dfrac{|h|^{n+1}}{(n+1)!}.$

If $\quad h > 0$, then $e^{\theta h} < e^h$, $\quad$ and $\quad 0 < R < \dfrac{h^{n+1}}{(n+1)!}e^h.$

Now
$$\lim_{n=\infty} \frac{h^{n+1}}{(n+1)!} = 0.$$

For we can write

$$\frac{h^{n+1}}{(n+1)!}=\frac{h}{1}\cdot\frac{h}{2}\cdot\frac{h}{3}\cdot\cdots\cdot\frac{h}{n}\cdot\frac{h}{n+1}.$$

No matter how large h may be numerically, since it is *fixed* and n is *variable*, these factors ultimately become small, and hence, in particular, from a definite point $n = m$ on

$$\frac{|h|}{n}<\frac{1}{2}, \qquad n \geqq m.$$

If we denote, then, the product of the first m factors, taken numerically, by C, and replace each of the subsequent factors by $\frac{1}{2}$, we shall have:

$$\left|\frac{h^{n+1}}{(n+1)!}\right|<C\left(\frac{1}{2}\right)^{n-m+1}.$$

The limit of this last expression is 0 when $n=\infty$, and consequently* $\lim\limits_{n=\infty} h^{n+1}/(n+1)! = 0$.

We have, then, $\lim\limits_{n=\infty} R = 0$ and hence, replacing h by x:

(1) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

The series converges and represents the function for all values of x.

If we set $x = 1$, we obtain the following rapidly converging series for the exponential base:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

Development of $\sin x$. To develop $\sin x$ we observe that

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0,\\ f'(x) &= \cos x, & f'(0) &= 1,\\ f''(x) &= -\sin x, & f''(0) &= 0,\\ f'''(x) &= -\cos x, & f'''(0) &= -1, \end{aligned}$$

and from this point on these values repeat themselves.

* We might have given a short proof of this relation by observing that $h^{n+1}/(n+1)!$ is the general term of a convergent series:

$$1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots.$$

It is not difficult to get a general expression for the n-th derivative, namely:

$$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right).$$

This formula obviously holds for $n = 1, 2, 3, 4$, and from that point on the right-hand member repeats itself, as it should.

Thus we find:

$$R = \frac{h^{n+1}}{(n+1)!} \sin\left(\theta h + \frac{(n+1)\pi}{2}\right).$$

The second factor is never greater than 1 numerically, and the first factor, as we have just seen, approaches 0 as its limit. Hence $\lim\limits_{n=\infty} R = 0$ and we have, replacing h by x:

(2) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

In a similar manner it is shown that

(3) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

EXERCISES

1. Show that e^x can be developed by Taylor's Theorem about any point a.

2. Obtain a general expression for the n-th derivative of $\cos x$ and hence prove the development (3).

3. Show that $\sin x$ and $\cos x$ can be developed by Taylor's Theorem about any point a.

24. Proof of the Binomial Theorem. Let

$$f(x) = x^m,$$

where m is any constant, integral, fractional, or incommensurable, positive or negative; and let $a = 1$.

Then $f(1) = 1$ and

$$\begin{aligned} f'(x) &= m x^{m-1}, & f'(1) &= m, \\ f''(x) &= m(m-1)x^{m-2}, & f''(1) &= m(m-1), \\ &\cdots\cdots & &\cdots\cdots \end{aligned}$$

$$f^{(n)}(x) = m(m-1)\cdots(m-n+1)x^{m-n},$$
$$f^{(n)}(1) = m(m-1)\cdots(m-n+1).$$

For the remainder R it is better here to employ the second form, (6), in § 22. Thus

$$R = \frac{(1-\theta)^n h^{n+1}}{n!}\cdot m(m-1)\cdots(m-n)(1+\theta h)^{m-n-1}.$$
$$= \frac{m(m-1)\cdots(m-n)}{n!}h^{n+1}\left(\frac{1-\theta}{1+\theta h}\right)^n(1+\theta h)^{m-1}.$$

The last factor remains finite, whatever the value of θ, provided $|h|<1$. For, since $0<\theta<1$,

$$1-|h|<1+\theta h<1+|h|,$$

and by Chap. VI, § 6:

$$(1+\theta h)^{m-1}<(1+|h|)^{m-1}, \qquad m>1;$$
$$(1+\theta h)^{m-1}<(1-|h|)^{m-1}, \qquad m<1.$$

The next to the last factor is always positive and less than unity, since $n>0$ and

$$0<\frac{1-\theta}{1+\theta h}<1.$$

Finally, the remaining expression is the general term of a series already shown to be convergent, namely, the following:

$$m(m-1)h^2+\frac{m(m-1)(m-2)}{2!}h^3+\frac{m(m-1)(m-2)}{3!}h^4+\cdots,$$

and hence it approaches 0 as its limit. It follows, then, that R approaches 0 and we have on replacing h by x:

(1) $$(1+x)^m = 1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\cdots.$$

This is the Binomial Theorem for negative and fractional exponents. When m is 0 or a positive integer, the series breaks off of itself with a finite number of terms and we have a polynomial, namely: $(1+x)^m$. In all other cases the series

converges when x is numerically less than 1 and represents the function $(1+x)^m$; and it diverges when x is numerically greater than 1.

EXERCISE

Show that, when $|a| > |b|$:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots$$

CHAPTER XV

PARTIAL DIFFERENTIATION

1. Functions of Several Variables. A function may depend on several independent variables. Thus the lateral area, S, of a cylinder of revolution is given by the formula

$$S = 2\pi rh,$$

and here r and h may, independently of each other, be any two positive numbers whatever. Again, the volume, u, of a rectangular parallelepiped is expressed by the equation:

$$u = xyz,$$

where x, y, z are the lengths of the three edges, and may, independently of each other, take on any three positive values.

If the number of independent variables is two, the function can be represented geometrically as a surface:

$$(1) \qquad z = f(x, y).$$

Such a function is said to be *continuous* at a point if the surface which represents it is continuous at the corresponding point. For example, the equation of a sphere with its centre at the origin:

$$(2) \qquad x^2 + y^2 + z^2 = a^2,$$

leads to a function, z, of the two independent variables, x and y:

$$(3) \qquad z = \sqrt{a^2 - x^2 - y^2}.$$

The surface represented by this function is the hemisphere above the (x, y)-plane. The equation of the hemisphere below this plane is

$$(4) \qquad z = -\sqrt{a^2 - x^2 - y^2}.$$

In the case of the functions (3) and (4), the point (x, y) is restricted to lie in the circle

$$x^2 + y^2 = a^2;$$

for, points (x, y) outside this circle make the radicand $a^2 - x^2 - y^2$ negative, and hence, for such, the function has no meaning.

2. Partial Derivatives. Let

$$(1) \qquad z = f(x, y)$$

be a function of the two independent variables, x and y. Let one of these variables, as y, be given a fixed value, $y = y_0$. Then z becomes a function of the single variable, x:

$$z = f(x, y_0).$$

This function of x can be differentiated with respect to x. The derivative is called the *partial derivative with respect to x*, and is written:

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x}.$$

(Read: "*D* x of z" or "*D* x of f.")

Similarly, x may be held fast, $x = x_0$, and thus z becomes a function of the single variable, y:

$$z = f(x_0, y).$$

This function of y can be differentiated with respect to y, and we have:

$$\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial f}{\partial y}.$$

If there were three independent variables, there would be still a third partial derivative, etc.

In distinction from the partial derivatives of a function of several variables the derivative of a function of a single variable is called a *total derivative.*

Example 1. Let $$z = e^x \cos y.$$

Then $$\frac{\partial z}{\partial x} = e^x \cos y \quad \text{and} \quad \frac{\partial z}{\partial y} = -e^x \sin y.$$

Example 2. Let $$u = xyz.$$

Then $$\frac{\partial u}{\partial x} = yz, \qquad \frac{\partial u}{\partial y} = xz, \qquad \frac{\partial u}{\partial z} = xy.$$

Example 3. Let z be given by the implicit equation (2), § 1. To find $\partial z/\partial x$ we hold y fast and differentiate the equation as it stands:

$$2x + 2z\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial z}{\partial x} = -\frac{x}{z}.$$

Other notations for $\dfrac{\partial f}{\partial x}$ are $f_x(x, y)$ and $f_1(x, y)$; for $\dfrac{\partial f}{\partial y}$ are $f_y(x, y)$ and $f_2(x, y)$, with the obvious extensions to functions of more than two variables.

EXERCISES

Find all the partial derivatives of each of the following functions.

1. $z = \sin xy$. *Ans.* $\dfrac{\partial z}{\partial x} = y \cos xy$; $\dfrac{\partial z}{\partial y} = x \cos xy$.

2. $z = ax^2 + 2bxy + cy^2$.

3. $z = x \log y$.

4. $z = \dfrac{x - y}{x + y}$.

5. $z = \dfrac{xy}{x^2 + y^2}$.

6. $u = x^3 + y^3 + z^3$.

7. $u = e^{xyz}$.

8. If $$pv = ap_0v_0T,$$
where a, p_0, v_0 are constants, find $\dfrac{\partial v}{\partial T}$.

9. If $$u = e^x \cos y \quad \text{and} \quad v = e^x \sin y,$$
show that $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

3. Geometric Interpretation. Tangent Plane and Normal Line. Geometrically, the meaning of the partial derivatives, in the case of a function z of but two independent variables, x and y:

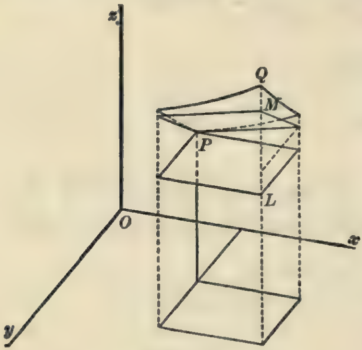


FIG. 133

$$(1) \qquad z = f(x, y),$$

is as follows. Holding y fast is equivalent to cutting the surface (1) by the plane $y = y_0$. The section is a plane curve:

$$(2) \quad z = f(x, y) \qquad y = y_0,$$

and $\frac{\partial z}{\partial x}$ is the slope of this curve. Similarly, $\frac{\partial z}{\partial y}$ is the slope of the plane curve

$$(3) \qquad z = f(x, y), \qquad x = x_0.$$

We thus have the slopes of two tangent lines to the surface (1) at the point (x_0, y_0, z_0), and hence we can readily determine the equation of the tangent plane through this point. For, the tangent plane at a point contains all the tangent lines at the point and is determined by any two of them. If, therefore, we write the equation of the tangent plane with undetermined coefficients in the form:

$$(4) \qquad z - z_0 = A(x - x_0) + B(y - y_0),$$

we have only to require that the slope of the line in which this plane is cut by the plane $y = y_0$, — *i.e.* the slope of the line

$$z - z_0 = A(x - x_0)$$

in the plane $y = y_0$, — have the same value as the slope of the curve (2) in this same plane at the point (x_0, y_0, z_0), or

$$A = \left(\frac{\partial z}{\partial x}\right)_0.$$

Similarly, the slope of the line in which the plane (4) is cut by the plane $x = x_0$, *i.e.* the slope of the line

$$z - z_0 = B(y - y_0)$$

in the plane $x = x_0$, —shall be the same as the slope of the curve (3) in the plane $x = x_0$ at the point (x_0, y_0, z_0), or

$$B = \left(\frac{\partial z}{\partial y}\right)_0.$$

Hence we infer that *the equation of the tangent plane to the surface* (1) *at the point* (x_0, y_0, z_0) *is*

$$(5) \qquad z - z_0 = \left(\frac{\partial z}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial z}{\partial y}\right)_0 (y - y_0).$$

The direction components of a normal to this plane are $\left(\frac{\partial z}{\partial x}\right)_0, \left(\frac{\partial z}{\partial y}\right)_0, -1$, and so the equations of the normal through the point (x_0, y_0, z_0) are:

$$(6) \qquad \frac{x - x_0}{\left(\frac{\partial z}{\partial x}\right)_0} = \frac{y - y_0}{\left(\frac{\partial z}{\partial y}\right)_0} = \frac{z - z_0}{-1}.$$

EXERCISES

Find the equations of the tangent plane and the normal to the following surfaces:

1. $z = \tan^{-1}\frac{y}{x}$. *Ans.* $y_0 x - x_0 y + (x_0^2 + y_0^2)(z - z_0) = 0$;

$$\frac{x - x_0}{y_0} = \frac{y - y_0}{-x_0} = \frac{z - z_0}{x_0^2 + y_0^2}.$$

2. $$z = ax^2 + by^2.$$

Ans. For the tangent plane: $z = 2ax_0 x + 2by_0 y - z_0$.

3. The sphere: $x^2 + y^2 + z^2 = a^2$

Ans. $x_0 x + y_0 y + z_0 z = a^2$; $\frac{x - x_0}{x_0} = \frac{y - y_0}{y_0} = \frac{z - z_0}{z_0}$.

4. The ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

5. The hyperboloid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$

6. Show that the surface

$$z = xy$$

is tangent to the x, y plane at the origin.

7. The sphere: $\qquad x^2+y^2+z^2=14$

and the ellipsoid: $\qquad 3x^2+2y^2+z^2=20$

intersect in the point $(-1, -2, 3)$. Find the angle at which they cut each other there. *Ans.* $23° 33'$.

8. What angle does the tangent plane of the ellipsoid in the preceding question make with the (x, y)-plane? *Ans.* $59° 2'$.

9. At what angle is the surface

$$z = 3xy^2 - 5x^2y - 7x + 3y$$

cut by the axis of x at the origin? *Ans.* $65° 41'$.

10. Show that the normal to the surface of Question 1, at any point where that surface is cut by the cylinder

$$x^2+y^2=1,$$

always makes an angle of 45° with the (x, y)-plane.

4. Derivatives of Higher Order. The partial derivative of a partial derivative is written as follows:

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right), \qquad \text{or} \qquad \frac{\partial^2 z}{\partial y\,\partial x}.$$

Consider Example 1, § 2. Here,

$$\frac{\partial^2 z}{\partial y\,\partial x}=\frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y,$$

and
$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial x}(-e^x \sin y) = -e^x \sin y.$$

Thus it appears that, in this case,

$$\frac{\partial^2 z}{\partial y\,\partial x}=\frac{\partial^2 z}{\partial x\,\partial y}.$$

This relation is true generally. *The order of differentiation is immaterial, or**

(1) $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

From the theorem stated for two independent variables and two differentations follows the theorem stated for any number of independent variables and any number of differentiations. Thus, if

$$u = f(x, y, z),$$

we have: $$\frac{\partial^2}{\partial y \partial x}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial z}\right), \quad \text{or} \quad \frac{\partial^3 u}{\partial y \partial x \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z}.$$

But $\dfrac{\partial^2 u}{\partial y \partial z} = \dfrac{\partial^2 u}{\partial z \partial y}$ and hence $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = \dfrac{\partial^3 u}{\partial x \partial z \partial y}$.

Since the first two differentiations can be interchanged, and the same is true of the last two, it is readily shown that all six possible orders lead to identical results.

EXERCISES

Verify the theorem in each of the following cases.

1. $$u = ax^2y + bxy^2 + cx^2y^2.$$

2. $u = x \cos xy$. 3. $u = x^y$.

4. If $$u = xyz,$$

compute all the partial derivatives of the second order: $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial x \partial y}$, etc., and verify the above theorem in the case of the cross-derivatives $\left(\dfrac{\partial^2 u}{\partial x \partial y}, \text{etc.}\right)$.

5. If $$u = e^x \cos y,$$

show that $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

* It is assumed that all the partial derivatives which enter are continuous functions. The proof of the theorem is omitted.

6. If
$$u = r^m \cos m\theta,$$
show that
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

7. The same, if
$$u = \frac{\sin m\theta}{r^m}.$$

8. If
$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}},$$
show that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

9. If
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$
then
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

10. If
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r},$$
then
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

5. Differentials. In the case of a function of a single variable, $y = f(x)$, the differential of the function, dy, was represented geometrically by the distance MQ, Fig. 33, from the level of P to the tangent. The differential of the independent variable, dx, and its increment, Δx, are equal.

Similarly, in the case of a function of two variables,

(1) $$z = f(x, y)$$

if we cut the surface (1) by the line $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, the distance LM, Fig. 133, measured along this line from the level of P up to the tangent plane,
$$z - z_0 = \left(\frac{\partial z}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial z}{\partial y}\right)_0 (y - y_0),$$
represents the differential, dz, of the function. Its value is

(2) $$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy;$$

for, the differentials of the independent variables are equal to the increments:

$$dx = \Delta x, \qquad dy = \Delta y.$$

If u is a function of x, y, z, then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz, \tag{3}$$

and similarly for any number of independent variables.

Equation (2), — and similarly (3), etc. — holds, not merely when (x, y) are the independent variables, but when x and y are functions either of a single variable, t, or of several variables, as (r, s), or (r, s, t), etc. To this fact is due the importance of differentials in the problem of the change of variables. But any adequate discussion of this subject must be postponed for a systematic treatment of Partial Differentiation.

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